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VISCID CONTRIBUTIONS TO THE HYDRODYNAMIC FLOWS AND HEAT TRANSFERS PAST A RISING SPHERICAL DEFORMING GAS BUBBLE

Radomir V. Ašković

Univ. Valenciennes - LME
Résidence Verley 11/09, 59300 Valenciennes , France
E-mail: radomir.askovic@univ-valenciennes.fr

Summary. *An analysis is made of the effect of the viscosity on the hydrodynamic flow fields and heat transfers past the interface of a spherical deforming gas bubble impulsively started at a constant velocity in a viscous liquid of large extent initially at rest. Exact solutions for the unsteady (outer and inner) flow fields and heat transfers within the boundary layers are obtained making appropriate scalings on the position, velocity, temperature and time variables in the Navier-Stokes and energy equations. These theoretical results apply to any slowly deforming fluid sphere, whatever the time-dependence of its radius, provided that the bubble retains its spherical shape, the internal circulation is complete, the flow separation is negligible and the Reynolds and Peclet numbers are large.*

Key words and phrases : *deforming gas bubble, viscous effect, heat transfer, scaling procedure.*

1 INTRODUCTION

Knowledge of the hydrodynamic flow fields past spherical objects such as bubbles or droplets moving in a viscous liquid is central in the underlying calculation of heat and mass transfers at the interface, or the calculation of the drag force acting on them as well. When the sphere radius is a time-dependent variable, one is faced with the calculation of the flow of a fluid over another deformable fluid, a problem that was already investigated by Hadamard [1] in his earliest work on this subject. The aim of this paper is to analyse the effect of viscosity on the unsteady hydrodynamic flow fields and heat transfers associated with an impulsively started translating bubble of a given gas, moving at a constant velocity in a surrounding viscous liquid of large extent at rest. Our calculations are based on the thin boundary layer approximation, according to a perturbative scheme valid under the condition of sufficiently large Reynolds numbers. Exact unsteady solutions of the both outer and inner boundary layers around a spherical interface have already been derived $[2]$, in the case where the radius R is kept fixed with time. However, a more frequent situation is encountered with deforming bubbles, where R turns out to be a time-dependent function R(t). It concerns for instance the expansion of compressed gas bubbles. In this work, we assume that the bubble retains its spherical shape as it moves. It has been shown $[3]$, that it holds even better since the prevailing (external) Reynolds number is moderate ($R_e \leq 500$). For instance, it is known from experimental observations (see $[4]$ and references therein) that air bubbles rising in water remain spherical provided the Reynolds number does not exceed 400. For larger values, a departure from sphericity occurs. Fortunately, in our calculations a bound $R_e \prec 400$ is large enough to ensure the validity of the boundary layer approximation since, as shown below, viscid contributions to the hydrodynamic flows past the bubble are of order $1/\sqrt{R_e}$ (see Eqs. (51) and (52) below) so they are negligible when R_e exceeds four hundred. A sufficiently large value of the interfacial tension σ is also needed to prevent modification of the shape by damping inertial effects. In fact, it has been known that a bubble remains nearly spherical even when the Reynolds number becomes large, provided that the Weber number $2R_0\rho_e U_\infty^2/\sigma$ remains small $[5]$. To place numbers on this condition, let us introduce the constant $k = (2\sigma/\rho_e\Gamma)^{1/2}$ which acts like a capillarity constant for the liquid, when calculated with the acceleration Γ experienced by the bubble (strictly speaking, k corresponds to the capillarity constant of the liquid when Γ is the gravitational acceleration g and σ its surface tension against his own vapor). The above mentioned condition is filfilled provided $R \leq k$. In the impulsive step of the motion, $R = R_0$ and Γ is obviously very large compared to the characteristic value U_∞^2/R_0. This yields an upper bound on the initial Weber number $2R_0\rho_e U_\infty^2/\sigma \ll 4$. In the rectilinear step of the motion, taken at a constant velocity, modifications of the shape can also occur by bubble oscillations. The latter are characterized by a minimum frequency given by $[6]$ $\omega_{\min} = (8\sigma/\rho_e R^3)^{1/2}$. Thus, along its motion, the shape of the bubble will remain spherical on average provided $U_\infty/R \ll \omega_{\min}$. This yields another condition on the (dynamic) Weber number $2R\rho_e U_\infty^2/\sigma \ll 16$. A further simplification is made through the assumption that there is no mass transfer at the interface. It is important to note that due to the variation of the bubble radius with time, it follows that the external and internal Navier-Stokes equations to solve remain implicitly non-linear even after the simplification of the convective terms.

2 DYNAMIC PART OF THE PROBLEM

2.1. Mathematical formulation of the unsteady problem.

Let us consider unsteady flows past a fluid sphere whose radius grows (or decreases) with a given time-dependence law R=R(t). The bubble is assumed to be impulsively started at time t=0 into a rectilinear motion at constant velocity U_∞ in a viscous incompressible liquid initialy at rest. For practical calculations, it is more convenient to consider the reversed situation where the bubble is at rest in a liquid with a velocity $-U_\infty$ at a large distance from it.

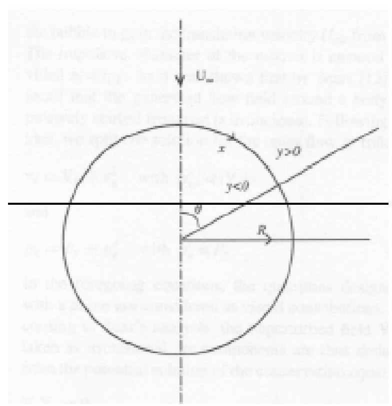

Figure 1. Curvilinear system of coordinates at the special interface.

In what follows, the external and internal fluids will be referenced by the subscripts e and i, respectively. Each of them is characterized by its specific gravity ρ, and its dynamic and kinematic viscosity, μ and $\nu = \mu/\rho$, respectively. For each of them also, the relevant Reynolds number is $R_e = 2RU_\infty / \nu$. In addition, two dimensionless numbers have to be considered [7] : the velocity ratio $A(t) = [dR/dt]/U_\infty$ and the scaling ratio $\gamma(t) = R(t)/R_0$, where R_0 denotes the initial value of the radius (i.e., $R_0 \equiv R(t=0)$).

We denote by $\vec{r} = (r,\theta)$ the position vector in the flow fields, with origin taken at the center of the bubble, where r is the magnitude of $\vec{r}$ and θ is the angle between $\vec{r}$ and the upstream axis of symmetry (Fig. 1). For convenience we will also make use of the notation $\vec{r} = r\vec{n}$ where $\vec{n}$ denotes a unit vector taken outward of the bubble surface. The conservation equation of mass and the Navier-Stokes equation of motion for the outer flow are:

$$\nabla \cdot \vec{v}_e = 0 \tag{1}$$

$$\frac{\partial \vec{v}_e}{\partial t} + \vec{v}_e \cdot \nabla \vec{v}_e = -\frac{1}{\rho_e} \nabla p_e + \nu_e \Delta \vec{v}_e \tag{2}$$

Eq. (1) holds since the external fluid is assumed to be incompressible. In Eq. (2) p_e is the pressure field implicitly including the body force due to gravity.

Similarly, the equations for the inner flow read as:

$$\nabla(\rho_i \vec{v}_i) + \frac{\partial \rho_i}{\partial t} = 0 \tag{3}$$

$$\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i \cdot \nabla \vec{v}_i = -\frac{1}{\rho_i} \nabla p_i + \nu_i \Delta \vec{v}_i + \frac{1}{\rho_i}(\varsigma + \frac{\mu}{3}) \nabla \nabla \cdot \vec{v}_i \tag{4}$$

In addition to the shear viscosity, the r.h.s. of the Navier-Stokes equation (4) contains a bulk viscosity term □ ς. The outer and inner flow fields $\vec{v}_e$ and $\vec{v}_i$ must obey the following interfacial condition:

$$J = \rho_i (\vec{v}_i - \vec{V}) \cdot \vec{n} = \rho_e (\vec{v}_e - \vec{V}) \cdot \vec{n} \tag{5}$$

where $\vec{V} = \vec{n} \cdot \left[dR/dt \right]$ and J is the radial mass flux across the interface,

Eqs. (3) and (4) no longer simplify as for the external case since the inlet fluid is obviously compressible. We firstly restrict our analysis to the case where the gas density within the bubble remains homogeneous, which means $\nabla \rho_i = 0$. This is a condition for internal mechanical equilibrium, which prevails provided the velocity deformation dR/dt is very small compared to the sound velocity. According to (5), the conservation equation (3) becomes:

$$\nabla \cdot \vec{v}_i = -\frac{d}{dt} \ln \rho_i = \frac{3}{R} (\frac{J}{\rho_i} + \frac{dR}{dt}) \tag{6}$$

Secondly we restrict to the case where the mass flux at the interface is nearly equal to zero namely $J \to 0$. It concerns for instance the sudden expansion (or depression) of a compressed (or depressed) gas bubble in a surrounding liquid with $p_e \neq p_i$, when the diffusion process at the interface is negligible, and therefore the mass content □ $\rho_i R^3$ remains nearly constant with time. In this case, Eq. (6) becomes:

$$\nabla \cdot \vec{v}_i = \frac{3}{R}\frac{dR}{dt} \tag{7}$$

The outer and inner flow fields then fulfill the same interfacial condition:

$$\lim_{r \to R} \vec{v}_e \cdot \vec{n} = \lim_{r \to R} \vec{v}_i \cdot \vec{n} = \frac{dR}{dt} \tag{8}$$

Moreover, since from Eq. (7) $\nabla \cdot \vec{v}_i$ is only a time-dependent function, the viscid contribution $\sim \nabla\nabla \cdot \vec{v}_i$ in the r.h.s. of Eq. (4) is identically zero.

Making use of the concept of Levich $[8]$ (1949) and Chao $[9]$ of a boundary layer over a non-deforming fluid interface, we develop in what follows a method based on the thin boundary layer approximation in order to analyse the two flow fields around a spherical gas bubble moving in a liquid. Let us denote by $\delta_{i,e}$ the thickness of thesse boundary layers for the inner and outer flow. A typical value for the evolution time of the flow fields within these layer sis given by $\tau_{i,e} \sim \delta_{i,e}^2 / \nu_{i,e}$. It must remain very small compared to the characteristic time of the translating motion $\tau_{i,e} \sim R/U_\infty$. This analysis provides a very simple condition of the thickness of the inner and outer boundary layers $\delta_{i,e}/R \sim \sqrt{2/\mathrm{Re}_{i,e}}$. It should be appreciated, however, that Moore $[10]$ and Harper and Moore $[11]$ have demonstrated that for a bubble under steady conditions, the thickness of the boundary layers changes from the front to the rear of the stagnation point.

Our aim is to analyse the development of the flow fields within these boundary layers for a bubble impulsively started from rest. Let t_0 be the time required for the bubble to gain the translation velocity U_∞ from rest. The impulsive character of the motion is ensured provided $t_0 \ll \tau_{i,e}$. As it was shown first by Sears $[12]$, we recall that the generated flow field around a body impulsively started from rest is irrotational. Following this idea, we split the solution for the outer flow as:

$$\vec{v}_e = \vec{V}_e + \vec{v}_e^{'} \quad \text{with} \quad \left|\vec{v}_e^{'}\right| \ll \left|\vec{V}_e\right| \tag{9}$$

$$p_e = P_e + p_e^{'} \quad \text{with} \quad p_e^{'} \ll P_e \tag{10}$$

In the foregoing equations, the quantities designated with a prime are considered as viscid contributions. According to Sears' analysis, the unperturbed field $\vec{V}_e$ is taken as irrotational. Its components are thus deduced from the potential solution of the conservation equation:

$$\nabla \cdot \vec{V}_e = 0\,, \tag{11}$$

while the pressure field P_e itself is a solution of the Euler equation:

$$\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e = -\frac{1}{\rho_e} \nabla P_e \tag{12}$$

For a non-deforming sphere, the tangential and radial components of $\vec{V}_e$, namely U_e and V_e, read:

$$U_e = U_\infty (1 + \frac{1}{2}\frac{R^3}{r^3}) \sin\theta \ , \quad r \geq R \tag{13}$$

$$V_e = -U_\infty (1 - \frac{R^3}{r^3}) \cos\theta \ , \quad r \geq R \tag{14}$$

In the case of a deforming bubble, with R=R(t), these equations still hold but they have to be supplemented with a purely radial solution of the continuity equation (11), thus proportional to $\vec{n}/r^2$, which corresponds to the purely radial external flow generated by the bubble deformation. Denoting it as $\vec{V}_{e_r}$ and noting from (14) that $\lim_{r \to R} V_e = 0$, it follows that it must fulfill the following interfacial condition:

$$\lim_{r \to R} \vec{V}_{e_r} = \vec{n} \frac{dR}{dt} \ . \tag{15}$$

Thus we obtain:

$$\vec{V}_{e_r} = \vec{n} (\frac{R}{r})^2 \frac{dR}{dt} , \tag{16}$$

so that:

$$V_e = -U_\infty (1 - \frac{R^3}{r^3}) \cos\theta + (\frac{R}{r})^2 \frac{dR}{dt} \ , \quad r \geq R \ . \tag{17}$$

Finally, according to decompositions (9) and (10), the outer viscid flow fields around a spherical deforming bubble fulfill the equations:

$$\nabla \cdot \vec{v}_e^{'} = 0 \tag{18}$$

$$\frac{\partial \vec{v}_e^{'}}{\partial t} + \vec{v}_e^{'} \cdot \nabla \vec{V}_e + \vec{V}_e \cdot \nabla \vec{v}_e^{'} = -\frac{1}{\rho_e} \nabla p_e^{'} + \nu_e \Delta \vec{v}_e^{'} \tag{20}$$

In Eq. (19), the term $\vec{v}_e^{'} \cdot \nabla \vec{v}_e^{'}$ has been discarded, since it is of second order.

Coming back to the internal flow, a similar analysis can be performed. As for the external flow, we split $\vec{v}_i$ and p_i as follows:

$$\vec{v}_i = \vec{V}_i + \vec{v}_i^{'} \quad \text{with} \quad \left|\vec{v}_i^{'}\right| \ll \left|\vec{V}_i\right| , \tag{20}$$

$$p_i = P_i + p_i^{'} \quad \text{with} \quad p_i^{'} \ll P_i^{'} \tag{21}$$

$\vec{V}_i$ corresponds to the circulatory motion of the inviscid fluid within the sphere due to the external irrotational flow $\vec{V}_e$. Its tangential and radial velocity components, namely U_i and V_i, were derived by Hill $[13]$:

$$U_i = -\frac{3}{2}U_\infty(1-2\frac{r^2}{R^2})\sin\theta, \quad r \le R, \tag{22}$$

$$V_i = \frac{3}{2}U_\infty(1-\frac{r^2}{R^2})\cos\theta, \qquad r \le R. \tag{23}$$

Since $\vec{V}_i$ is rotational, it fulfills the conservation equation:

$$\nabla \cdot \vec{V}_i = 0 \tag{24}$$

It should be appreciated that when the bubble starts to move, if the inlet flow is assumed to be at rest, i twill take a finite time, of order t_0, to reach the Hill vortex. The condition $t_0 \ll \tau_i$, however, ensures that the Hill solution applies practically from the earlier stage of the viscid flow field expansions (i.e., from rest) in our calculations.

Solutions (22) and (23) still apply to the case of a uniformly deforming bubble, but as for the outer flow, they have to be supplemented with a purely radial solution $\vec{V}_{i_r}$ of Eq. (7) in order to account for the purely radial internal flow generated by the bubble deformation. Once again taking into account the interfacial condition:

$$\lim_{r\to R}\vec{V}_{i_r} = \vec{n}\frac{dR}{dt} \tag{25}$$

we obtain (since $\nabla \cdot \vec{r} = 3$):

$$\vec{V}_{i_r} = \vec{n}(\frac{r}{R})\frac{dR}{dt} \tag{26}$$

Finally, the radial component of the inner inviscid flow field reads:

$$V_i = \frac{3}{2}U_\infty(1-\frac{r^2}{R^2})\cos\theta + \frac{r}{R}\frac{dR}{dt} \tag{27}$$

In comparison to the external case, one obtains the following equations for the inner flow:

$$\nabla \cdot \vec{v}_i^{'} = 0 \tag{28}$$

$$\frac{\partial \vec{v}_i^{'}}{\partial t} + \vec{v}_i^{'} \cdot \nabla\vec{V}_i + \vec{V}_i \cdot \nabla\vec{v}_i^{'} = -\frac{1}{\rho_i}\nabla p_i^{'} + \nu_i \Delta\vec{v}_i^{'} \tag{29}$$

2.3. Perturbative method of analysis.

Working with the (r,θ) system of coordinates, one has $\nabla = ((1/r)\,\partial/\partial\theta, \partial/\partial r)$, so that the continuity equation (18) and the projection of the equation of motion (19) along the tangential direction read:

$$\frac{1}{r}\frac{\partial}{\partial\theta}(u'_e \sin\theta) + \sin\theta \frac{\partial v'_e}{\partial r} = 0, \tag{30}$$

$$\frac{\partial u'_e}{\partial t} + u'_e \frac{1}{r}\frac{\partial U_e}{\partial\theta} + v'_e \frac{\partial U_e}{\partial r} + U_e \frac{1}{r}\frac{\partial u'_e}{\partial\theta} + V_e \frac{\partial u'_e}{\partial r} = -\frac{1}{\rho_e}\frac{1}{r}\frac{\partial p'_e}{\partial\theta} + \nu_e \Delta u'_e, \tag{31}$$

$u_e^{'}$ and $v_e^{'}$ denote the tangential and radial components of the flow field $\vec{v}_e^{\,'}$, respectively. Following a procedure suggested by Boltze in his study of boundary layers over a body of revolution, we now introduce a curvilinear system of coordinates $|14|$, as shown in Fig. 1. We denote by $x = R\theta$ the arc length measured along any meridian from the front stagnation point, and y the coordinate normal to the bubble surface, taken outward as positive defined according to r=R(t)+y. In the thin boundary layer approximation, valid at large Reynolds numbers, one obtains $|y| \ll R$, since $|y| \sim \delta_{i,e}$. Then, Eq. (31) can be simplified as follows. First, from the above definition of y, one has:

$$\left(\frac{\partial}{\partial t}\right)_r \equiv \left(\frac{\partial}{\partial t}\right)_y + \left(\frac{\partial y}{\partial t}\right)_r \left(\frac{\partial}{\partial y}\right)_t = \left(\frac{\partial}{\partial t}\right)_y - \frac{dR}{dt}\left(\frac{\partial}{\partial y}\right)_t. \tag{32}$$

Thus, transforming from (r,θ) to (y,θ) makes the contribution $\sim dR/dt$ within V_e (cf. Eq. (17)) to be cancelled when introduced in the equation of motion (31). Next, according to the usual method of perturbation, the following approximations are made:

$$\frac{U_e}{U_\infty} \approx \frac{3}{2}\left(1-\frac{y}{R}\right)\sin\theta\ , \quad y \ge 0, \tag{33}$$

$$\frac{V_e}{U_\infty} \approx -3\frac{y}{R}\cos\theta + \mathrm{A}\left(1-2\frac{y}{R}\right)\ , \quad y \ge 0. \tag{34}$$

We obtain

$$\frac{1}{R}\frac{\partial}{\partial\theta}(u_e^{'}\sin\theta)+\sin\theta\frac{\partial v_e^{'}}{\partial y}=0 \tag{35}$$

$$\frac{\partial u_e^{'}}{\partial t}+\frac{3}{2}\frac{U_\infty}{R}(u_e^{'}\cos\theta+\frac{\partial u_e^{'}}{\partial\theta}\sin\theta-2y\frac{\partial u_e^{'}}{\partial y}\cos\theta)$$
$$-2\frac{U_\infty}{R}\mathrm{A}y\frac{\partial u_e^{'}}{\partial y}=\nu_e\frac{\partial^2 u_e^{'}}{\partial y^2} \tag{36}$$

In Eq. (36), the time derivative is now implicitly taken at a constant y value. The term proportional to $v_e^{'}\left[\partial U_e/\partial r\right]$ in the left-hand side of Eq. (31) has been discarded since, as shown below, $v_e^{'}$ is of order $\left[y/R\right]u_e^{'}$ in the boundary layer, i.e., of a second order. On the right-hand side of Eq. (31) the contribution of the pressure term can be neglected. This simplification is based on the following dimensional analysis. As we are concerned with viscid contributions, it appears meaningful to scale $u_e^{'}$ with ν_e/δ_e, t with $\tau_e=\delta_e^2/\nu_e$, $p_e^{'}$ with $\rho_e(\nu_e/\delta_e)^2$ and finally r by R. It turns that the ratio of $1/\rho_e(1/r)\partial p_e^{'}/\partial\theta$ over $\partial u_e^{'}/\partial t$ is of order δ_e/R. For a similar reason, the viscous term □ $\nu_e\Delta u_e^{'}$ is restricted to $\nu_e(\partial^2 u_e^{'}/\partial y^2)$ since the contribution $\nu_e(\partial^2 u_e^{'}/\partial x^2)$ is of an order $(\delta_e/R)^2$ smaller. Finally, it should be noticed that with these considerations, Eq. (36) only involves the tangential component of $\vec{v}_e$, while the radial one is derived by integration of the continuity equation (35).

Very similar conclusions apply to the internal case. At first order we obtain:

$$\frac{U_i}{U_\infty}\approx\frac{3}{2}(1+4\frac{y}{R})\sin\theta\ ,\quad y\le 0, \tag{37}$$

$$\frac{V_i}{U_\infty}\approx-3\frac{y}{R}\cos\theta+(1+\frac{y}{R})\mathrm{A}\ ,\quad y\le 0 \tag{38}$$

and then:

$$\frac{1}{R}\frac{\partial}{\partial\theta}(u_i^{'}\sin\theta)+\sin\theta\frac{\partial v_i^{'}}{\partial y}=0 \tag{39}$$

$$\frac{\partial u_i^{'}}{\partial t}+\frac{3}{2}\frac{U_\infty}{R}(u_i^{'}\cos\theta+\frac{\partial u_i^{'}}{\partial\theta}\sin\theta-2y\frac{\partial u_i^{'}}{\partial y}\cos\theta)$$
$$+\frac{U_\infty}{R}\mathrm{A}y\frac{\partial u_i^{'}}{\partial y}=\nu_i\frac{\partial^2 u_i^{'}}{\partial y^2} \tag{40}$$

2.3. Conditions at the interface.

The foregoing equations have to be solved with the following conditions at the interface, valid when $y \to 0$:

$$(u_e^{'})_{y=0} = (u_i^{'})_{y=0} \tag{41}$$

This is the usual « non-slip » condition at the interface, namely:

$$(u_e)_{y=0} = (u_i)_{y=0} \tag{42}$$

Since $(U_e)_{y=0} = (U_i)_{y=0}$, the condition (41) follows immediately.

A second condition emerges from the requirement of continuity of the shear stress, i.e.:

$$\mu_e \left[\frac{\partial u_e}{\partial r} - \frac{u_e}{r} \right]_{r=R} = \mu_i \left[\frac{\partial u_i}{\partial r} - \frac{u_i}{r} \right]_{r=R} \tag{43}$$

Since $u_e^{'} = u_e - U_e$ and $u_i^{'} = u_i - U_i$, it follows from the potential solution (33) and the Hill solution (38) that:

$$\mu_e \left(\frac{\partial u_e^{'}}{\partial y}\right)_{y=0} - \mu_i \left(\frac{\partial u_i^{'}}{\partial y}\right)_{y=0} = \frac{3}{2} \frac{U_\infty}{R(t)} (2\mu_e + 3\mu_i) \sin\theta \, . \tag{44}$$

Similarly, since $v_e^{'} = v_e - V_e$ and $v_i^{'} = v_i - V_i$, it follows, taking into account (15) and (25), that the radial components of the outer and inner flow fields must obey the interfacial conditions:

$$(v_e^{'})_{y=0} = (v_i^{'})_{y=0} = 0 \tag{45}$$

2.4. Determination of the tangential and radial velocity components of both outer and inner flows.

First, by introducing the dimensionless variables:

$$\tilde{u}_e = \frac{u_e^{'}}{U_\infty} \; , \quad \tilde{u}_i = \frac{u_i^{'}}{U_\infty} \tag{46}$$

$$\tilde{y} = \frac{y}{R_0} \tag{47}$$

then by defining the appropriate scaled variables as:

$$\bar{u}_e = \gamma^{1/2} \tilde{u}_e \, , \quad \bar{u}_i = \gamma^{1/2} \tilde{u}_i \tag{48}$$

$$\bar{y} = \gamma^{-1/2} \tilde{y} \, , \tag{49}$$

we found recently [15] the solutions of the equations (36) and (40). It remains to complete this procedure by introducing also the dimensioless time variable according to:

$$\frac{d\tau}{dt} = \frac{U_\infty}{\gamma R_0}$$

by using the scaling ratio : $\gamma(t) = R(t)/R_0$ and the identity : $\frac{R_0}{U_\infty}\frac{d\gamma}{dt} \equiv \mathrm{A} = \frac{dR/dt}{U_\infty}$,

whence:

$$\tau(t) = U_\infty \int_0^t \frac{dt'}{R(t')} \tag{50}$$

It should be noticed that since the integrant is positive, (50) guarantees that the transformation $\tau \leftrightarrow t$ is invertible:

So, making use of the notation $\beta = \varsigma / \sin^4\theta$, the scaled tangential velocity components of both the outer and inner flows are [15] now:

$$\bar{u}_e = -3\frac{2\mu_e + 3\mu_i}{\mu_e (Re_{e0})^{1/2} + \mu_i (Re_{i0})^{1/2}} \times \beta^{1/2} \sin\theta \,\mathrm{ierfc}\, Z_e, \tag{51}$$

$$\bar{u}_i = -3\frac{2\mu_e + 3\mu_i}{\mu_e (Re_{e0})^{1/2} + \mu_i (Re_{i0})^{1/2}} \times \beta^{1/2} \sin\theta \,\mathrm{ierfc}\, Z_i, \tag{52}$$

in which the $i^n erfcZ$ functions (here with n=1) stand for repeated integrals of the error function of Gauss [16], where:

$$\zeta = \tfrac{4}{3}\left[(f - \cos\theta) - \tfrac{1}{3}(f^3 - \cos^3\theta)\right] \tag{53}$$

$$f = \frac{(1+\cos\theta) - (1-\cos\theta)\exp(-3\tau)}{(1+\cos\theta) + (1-\cos\theta)\exp(-3\tau)}. \tag{54}$$

The arguments Z_e and Z_i are defined as:

$$Z_e = \frac{\bar{y}}{2}\left(\frac{R_{ee_o}}{\beta}\right)^{1/2} \quad \text{and} \quad Z_i = \frac{|\bar{y}|}{2}\left(\frac{R_{ei_0}}{\beta}\right)^{1/2}, \tag{55}$$

where

$$R_{ee_0} = 2R_0 U_\infty / \nu_e \text{ and } R_{ei_0} = 2R_0 U_\infty / \nu_i \tag{56}$$

The radial velocity compnents of both outer and inner flows are deduced by integration of the continuity equations (35) and (39), taking into account of the conditions:

$$\overline{v}_{\mathrm{e}} = 2\frac{(2\mu_{\mathrm{e}}+3\mu_{\mathrm{i}})}{\mu_{\mathrm{e}}(Re_{\mathrm{e}0})^{1/2}+\mu_{\mathrm{i}}(Re_{\mathrm{i}0})^{1/2}}\frac{1}{\sqrt{Re_{\mathrm{e}0}}} \times\left\{-6\beta\cos\theta\left[\mathrm{i}^2\operatorname{erfc} Z_{\mathrm{e}}-\frac{1}{4}\right]+\operatorname{erf} Z_{\mathrm{e}} \times\left[1-3\beta\cos\theta-\left(\frac{1+f}{1+\cos\theta}\right)^4 \mathrm{e}^{-6\tau}\right]\right\} \tag{57}$$

$$\overline{v}_{\mathrm{i}} = -2\frac{(2\mu_{\mathrm{e}}+3\mu_{\mathrm{i}})}{\mu_{\mathrm{e}}(Re_{\mathrm{e}0})^{1/2}+\mu_{\mathrm{i}}(Re_{\mathrm{i}0})^{1/2}}\frac{1}{\sqrt{Re_{\mathrm{i}0}}} \times\left\{-6\beta\cos\theta\left[\mathrm{i}^2\operatorname{erfc} Z_{\mathrm{i}}-\frac{1}{4}\right]+\operatorname{erf} Z_{\mathrm{i}} \times\left[1-3\beta\cos\theta-\left(\frac{1+f}{1+\cos\theta}\right)^4 \mathrm{e}^{-6\tau}\right]\right\}. \tag{58}$$

Solutions (51), (52), (57) and (58) for the tangential and radial components of the flow fields are analogous to those derived $[2]$ in the case of a non-deforming bubble. It should be appreciated, however, that they involve an implicite time dependence, as the quantities $\overline{u}_e$, $\overline{u}_i$, $\overline{v}_e$ and $\overline{v}_i$ are now scaled by the dimensionless ratio $\gamma = R(t)/R_0$, according to laws (48). This time-dependence is also contained within the space variable $\overline{y}$ itself, scaled according to (49). They also exhibit an explicit time-dependence through the dimensionless time τ defined by (50), which appears in the exponential $e^{-6\tau}$ in Eqs. (57) and (58), and which is contained indirectly via ς within β, according to Eqs. (53) and (54). The above-mentioned solutions are valid whatever the law of deformation of the bubble radius with time, provided the condition $A = (dR/dt)/U_\infty \ll 1$ applies.

It should be appreciated that in the opposite limit $A \gg 1$, the equations of motion no longer have non-zero solutions compatible with the interfacial conditions, except the inviscid solutions (16) and (26) which correspond to a purely radial flow. This physically corresponds to the fact that a purely radial motion no longer involves shear effect, so that the viscous effect in that case is only contained within the bulk viscosity term in the Navier-Stokes equation. Finally, in the intermediate regime $A \sim 1$, terms proportional to A cannot be neglected. At present, we have not been able to solve these equations exactly, but we will try to do it by expansions of the unknown functions in successive powers of A.

3 ON THE TRANSIENT HEAT TRANSFER

3.1. Problem statement and major assumptions

We consider a fluid sphere of a given time-depending radius R(t), initially at a uniform and constant temperature T_0 and impulsively started at time t=0 into a rectilinear motion at constant velocity U_∞ in another fluid of infinite extent. We imagine that at time t=0 the temperature of the continious phase fluid undergoes a step change from T_0 to T_∞. It is desired to examine the transfer response behavior of the thermal boundary layers both inside and outside of the growing fluid sphere. While such idealized conditions can hardly be realized in practice, the results are useful in that they not only bring forth the salient features of the problem but also can be generalized to accommodate those instances where T_0 and T_∞ are prescribed functions of time.

Following the usual procedure of making order of magnitude estimates of the various terms in the governing conservation equations, we establish that the energy equations for the thermal boundary layers are:

$$\frac{\partial T_e}{\partial t} + v_e\frac{\partial T_e}{\partial y} + \frac{u_e}{R}\frac{\partial T_e}{\partial \theta} = a_e\frac{\partial^2 T_e}{\partial y^2}, \quad y > 0$$

$$\frac{\partial T_i}{\partial t} + v_i\frac{\partial T_i}{\partial y} + \frac{u_i}{R}\frac{\partial T_i}{\partial \theta} = a_i\frac{\partial^2 T_i}{\partial y^2}, \quad y < 0.$$

(59) and (60)

with the corresponding initial, boundary and interfacial conditions as follows:

$$\begin{array}{ll} (y>0) & (y<0) \\ T_e(y,\theta,0) = T_\infty \quad (e) & T_i(y,\theta,0) = T_0 \quad (i) \\ T_e(\infty,\theta,t) = T_\infty \quad (e) & T_i(-\infty,\theta,t) = T_0 \quad (i) \end{array}$$

$$\frac{\partial T_{e,i}}{\partial \theta}(y,0,t) = \frac{\partial T_{e,i}}{\partial \theta}(y,\pi,t) = 0, \qquad (i,e)$$

(61)

$$T_e(0,\theta,t) = T_i(0,\theta,t)$$

$$\lambda_e\frac{\partial T_e}{\partial y}(0,\theta,t) = \lambda_i\frac{\partial T_i}{\partial y}(0,\theta,t).$$

Taking into account that all the coefficients (u_e, v_e) and (u_i, v_i) are found preliminary after (9), (20), (51), (52), (57), (58), the only possible integration of the preceeding equations (59) and (60) has to be carried out numerically.

For a moment, we are going to analyse the thermal problem by using the following assumptions:

1. Fully developed internal circulation. Winnikov and Chao [18] have experimentally demonstrated that, in highly purified systems, moving droplets invariably

exhibit internal circulation. It is usually of such an extent that the wake becomes relatively insignificant .

2. Inviscid flow fields. Since only large Reynolds number is of interest, the inviscid approximation is valid. This is particularly true if the internal circulation is vigorous. Accordingly, the external flow is irrotational and the internal field is that of Hill's spherical vortex. Generally speaking, the viscous effect is small when the Reynolds number exceeds two or three hundred. It may be of interest to note that if the hydrodynamic boundary layers are developing simultaneously with the thermal boundary layers, the inviscid approximation is even better [19].

3. Thin boundary layers. Under the condition of large Peclet numbers, the boundary layers are thin except the region close to the rear stagnation.

4. Constant properties and negligible dissipation. These are the usual assumptions and are known to be valid except that the change of viscosity with temperature may deserve some consideration in certain instances. In view of the inviscid approximation adopted in this analysis, refinements to account for such effect may not be justified.

3.2. Mathematical analysis. Scaled energy equations.

So, taking into account of these assumptions, we establish that the energy equations for the thermal boundary layers are:

$$\frac{\partial T_{e1}}{\partial t}+V_e\frac{\partial T_{e1}}{\partial y}+\frac{U_e}{R}\frac{\partial T_{e1}}{\partial\theta}=a_e\frac{\partial^2 T_{e1}}{\partial y^2}\ ,\quad y\succ 0\,, \tag{59'}$$

$$\frac{\partial T_{i1}}{\partial t}+V_i\frac{\partial T_{i1}}{\partial y}+\frac{U_i}{R}\frac{\partial T_{i1}}{\partial\theta}=a_i\frac{\partial^2 T_{i1}}{\partial y^2}\ ,\quad y\prec 0\,. \tag{60'}$$

where the functions U_e, V_e, U_i, V_i are already found (33), (34), (37), (38).

Now, to solve the equations (59') et (60'), with the corresponding conditions (61), we can use the same type of procedure as in the dynamic part of the problem.

Let us first introduce the dimensionless variables:

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$$\tilde{T}_{e1}=\frac{T_{e1}-T_\infty}{T_0-T_\infty}\ ,\quad \tilde{T}_{i1}=\frac{T_{i1}-T_\infty}{T_0-T_\infty}\ ; \tag{62}$$

$$\tilde{y}=\frac{y}{R_0} \tag{63}$$

With these notations, the energy equations (59') and (60') read as:

$$\frac{R_0}{U_\infty}\frac{\partial \tilde{T}_{e1}}{\partial t}+\gamma^{-1}(-3\cos\theta\,\tilde{y}\frac{\partial \tilde{T}_{e1}}{\partial \tilde{y}}+\frac{3}{2}\sin\theta\frac{\partial \tilde{T}_{el}}{\partial \theta})-2\gamma^{-1}\mathrm{A}\tilde{y}\frac{\partial \tilde{T}_{e1}}{\partial \tilde{y}}=\frac{2}{P_{ee_0}}\frac{\partial^2 \tilde{T}_{e1}}{\partial \tilde{y}^2} \quad (64)$$

$$\frac{R_0}{U_\infty}\frac{\partial \tilde{T}_{i1}}{\partial t}+\gamma^{-1}(-3\cos\theta\,\tilde{y}\frac{\partial \tilde{T}_{i1}}{\partial \tilde{y}}+\frac{3}{2}\sin\theta\frac{\partial \tilde{T}_{i1}}{\partial \theta})+\gamma^{-1}\mathrm{A}\tilde{y}\frac{\partial \tilde{T}_{i1}}{\partial \tilde{y}}=\frac{2}{P_{ei_0}}\frac{\partial^2 \tilde{T}_{i1}}{\partial \tilde{y}^2} \quad (65)$$

In these equations, the time derivative is now taken at a constant $\tilde{y}$ value, γ^{-1} is the inverse of the scaling ratio $\gamma(t)=R(t)/R_0$ and P_{ee_0} and P_{ei_0} are the outer and inner Peclet numbers at time t=0 :

$$P_{ee_0}=\frac{2R_0U_\infty}{a_e}\ , \qquad P_{ei_0}=\frac{2R_0U_\infty}{a_i} \quad (66)$$

At this stage, we proceed to a time scaling on the variables $\tilde{y}$, $\tilde{T}_{e1}$ and $\tilde{T}_{i1}$, according to the following laws:

$$\tilde{y}=\gamma^a\bar{y} \quad (67)$$

$$\tilde{T}_{e1}=\gamma^b\bar{T}_{e1}\ , \qquad \tilde{T}_{i1}=\gamma^b\bar{T}_{i1} \quad (68)$$

The time derivative is then transformed as follows :

$$(\frac{\partial \tilde{T}_{e,i1}}{\partial t})_{\tilde{y}}=\gamma^b(\frac{\partial \bar{T}_{e,i1}}{\partial t})_{\tilde{y}}+b\gamma^{b-1}\frac{d\gamma}{dt}\bar{T}_{e,i1} \quad (69)$$

$$(\frac{\partial \bar{T}_{e,i1}}{\partial t})_{\tilde{y}}=(\frac{\partial \bar{T}_{e,i1}}{\partial t})_{\bar{y}}+(\frac{\partial \bar{T}_{e,i1}}{\partial \bar{y}})_t(\frac{\partial \bar{y}}{\partial t})_{\tilde{y}} \quad (70)$$

At $\tilde{y}$ kept fixed, from (67) we obtain:

$$0=a\gamma^{a-1}\frac{d\gamma}{dt}\bar{y}+\gamma^a(\frac{\partial \bar{y}}{\partial t})_{\tilde{y}}$$

$$(\frac{\partial \bar{y}}{\partial t})_{\tilde{y}}=-a\gamma^{-1}\frac{d\gamma}{dt}\bar{y}$$

then from (70):

$$(\frac{\partial \bar{T}_{e,i1}}{\partial t})_{\tilde{y}}=(\frac{\partial \bar{T}_{e,i1}}{\partial t})_{\bar{y}}-a\gamma^{-1}\frac{d\gamma}{dt}\bar{y}(\frac{\partial \bar{T}_{e,i1}}{\partial \bar{y}})_t\,.$$

so that finally from (69):

$$(\frac{\partial \tilde{T}_{e,i1}}{\partial t})_{\tilde{y}}=\gamma^b(\frac{\partial \bar{T}_{e,i1}}{\partial t})_{\bar{y}}-a\gamma^{b-1}\frac{d\gamma}{dt}\bar{y}(\frac{\partial \bar{T}_{e,i1}}{\partial \bar{y}})_t+b\gamma^{b-1}\frac{d\gamma}{dt}\bar{T}_{e,i1} \quad (71)$$

Making use of the foregoing transformations, after straightforward algebra, the energy equations (64) and (65) become:

$$\frac{R_0}{U_\infty}\left[\gamma(\frac{\partial\overline{T}_{e1}}{\partial t})_{\overline{y}}-a\frac{d\gamma}{dt}\overline{y}(\frac{\partial\overline{T}_{e1}}{\partial\overline{y}})_t+b\frac{d\gamma}{dt}\overline{T}_{e1}\right]+(-3\cos\theta\overline{y}\frac{\partial\overline{T}_{e1}}{\partial\overline{y}}+\frac{3}{2}\sin\theta\frac{\partial\overline{T}_{e1}}{\partial\theta})$$
$$-2\mathrm{A}\overline{y}\frac{\partial\overline{T}_{e1}}{\partial\overline{y}}=\frac{2}{P_{ee_0}}\frac{\gamma}{\gamma^{2a}}\frac{\partial^2\overline{T}_{e1}}{\partial\overline{y}^2} \tag{72}$$

$$\frac{R}{U_\infty}\left[\gamma(\frac{\partial\overline{T}_{i1}}{\partial t})_{\overline{y}}-a\frac{d\gamma}{dt}\overline{y}(\frac{\partial\overline{T}_{i1}}{\partial\overline{y}})_t+b\frac{d\gamma}{dt}\overline{T}_{i1}\right]+(-3\cos\theta\overline{y}\frac{\partial\overline{T}_{i1}}{\partial\overline{y}}+\frac{3}{2}\sin\theta\frac{\partial\overline{T}_{i1}}{\partial\theta})$$
$$+\mathrm{A}\overline{y}\frac{\partial\overline{T}_{i1}}{\partial\overline{y}}=\frac{2}{P_{ei_0}}\frac{\gamma}{\gamma^{2a}}\frac{\partial^2\overline{T}_{i1}}{\partial\overline{y}^2} \tag{73}$$

It remains to complete our analysis by introducing the dimensionless time variable as follows :

$$\frac{d\tau}{dt}=\frac{a_e}{\gamma R_0^2}\,, \tag{74}$$

whence:

$$\tau=\frac{a_e}{R_0^2}\int_0^t\frac{dt'}{\gamma(t')} \tag{75}$$

It can be noticed that, since the integrant is positive, from (75) we see that the transformation $\tau\leftrightarrow t$ is also invertible.

So, taking into account again the identity : $\frac{R_0}{U_\infty}\frac{d\gamma}{dt}\equiv\mathrm{A}=\frac{dR/dt}{U_\infty}$, the scaled energy equations (72) and (73) become:

$$(\frac{\partial\overline{T}_{e1}}{\partial\tau})_{\overline{y}}-\frac{3}{2}P_{ee_0}\cos\theta\overline{y}\frac{\partial\overline{T}_{e1}}{\partial\overline{y}}+\frac{3}{4}P_{ee_0}\sin\theta\frac{\partial\overline{T}_{e1}}{\partial\theta}$$
$$+\mathrm{A}\frac{1}{2}P_{ee_0}\left[-(a+2)\overline{y}(\frac{\partial\overline{T}_{e1}}{\partial\overline{y}})_\tau+b\overline{T}_{e1}\right]=\frac{\gamma}{\gamma^{2a}}\frac{\partial^2\overline{T}_{e1}}{\partial\overline{y}^2} \tag{76}$$

$$(\frac{\partial\overline{T}_{i1}}{\partial\tau})_{\overline{y}}-\frac{3}{2}P_{ee_0}\cos\theta\overline{y}\frac{\partial\overline{T}_{i1}}{\partial\overline{y}}+\frac{3}{4}P_{ee_0}\sin\theta\frac{\partial\overline{T}_{i1}}{\partial\theta}$$
$$+\mathrm{A}\frac{1}{2}P_{ee_0}\left[-(a-1)\overline{y}(\frac{\partial\overline{T}_{i1}}{\partial\overline{y}})_\tau+b\overline{T}_{i1}\right]=\frac{\gamma}{\gamma^{2a}}\frac{\partial^2\overline{T}_{i1}}{\partial\overline{y}^2} \tag{77}$$

These equations have to be solved with the conditions (61) which, for the scaled energy equations, read:

$$(y\succ 0)$$
$$T_{e1}(y,\theta,0)=T_\infty\rightarrow\rightarrow\tilde{T}_{e1}(\tilde{y},\theta,0)=0\rightarrow\rightarrow\overline{T}_{e1}(\overline{y},\theta,0)=0$$

$$T_{e1}(\infty,\theta,t)=T_{\infty}\rightarrow\rightarrow\tilde{T}_{e1}(\infty,\theta,t)=0\rightarrow\rightarrow\overline{T}_{e1}(\infty,\theta,\tau)=0 \tag{78}$$

$(y \prec 0)$

$$T_{i1}(y,\theta,0)=T_{0}\rightarrow\rightarrow\tilde{T}_{i1}(\tilde{y},\theta,0)=1\rightarrow\rightarrow\overline{T}_{i1}(\overline{y},\theta,0)=\gamma^{-b}$$

$$T_{i1}(-\infty,\theta,t)=T_{0}\rightarrow\rightarrow\tilde{T}_{i1}(-\infty,\theta,t)=1\rightarrow\rightarrow\overline{T}_{i1}(-\infty,\theta,\tau)=\gamma^{-b} \quad 79)$$

$$T_{e1}(0,\theta,t)=T_{i1}(0,\theta,t)\rightarrow\rightarrow\tilde{T}_{e1}(0,\theta,t)=\tilde{T}_{i1}(0,\theta,t)\rightarrow\rightarrow\overline{T}_{e1}(0,\theta,\tau)=\overline{T}_{i1}(0,\theta,\tau)$$

$$\Omega\frac{\partial\overline{T}_{e1}}{\partial\overline{y}}(0,\theta,\tau)=\frac{\partial\overline{T}_{i1}}{\partial\overline{y}}(0,\theta,\tau) \tag{80}$$

where

$$\Omega=\frac{\lambda_{e}}{\lambda_{i}}\left(\frac{a_{i}}{a_{e}}\right)^{1/2}$$

In what follows, we will restrict ourselves to the case of a bubble with a slow deforming velocity, i.e., $\mathrm{A} \ll 1$, which allows to neglect the terms proportional to A in the preceeding equations. Next, to complete our analysis, we will chose the exponents in the scaling relations. Namely, we take a=1/2 in order that the r.h.s. of the above equations involve only the initial Peclet number P_{ee_0}. Finally, we retain b=0 in order that the both initial and boundary conditions of the inner flow (79) become equal to unity.

With these prescriptions, the scaled variables are now defined as :

$\overline{T}_{e1}=\tilde{T}_{e1}$, $\overline{T}_{i1}=\tilde{T}_{i1}$.

$\overline{y}=\gamma^{-1/2}\tilde{y}$.

The governing conservation equations then can be written as:

$$\frac{\partial\overline{T}_{e1}}{\partial\tau}-\frac{3}{2}P_{ee_0}\cos\theta\,\overline{y}\,\frac{\partial\overline{T}_{e1}}{\partial\overline{y}}+\frac{3}{4}P_{ee_0}\sin\theta\frac{\partial\overline{T}_{e1}}{\partial\theta}=\frac{\partial^{2}\overline{T}_{e1}}{\partial\overline{y}^{2}} \tag{81}$$

$$\frac{\partial\overline{T}_{i1}}{\partial\tau}-\frac{3}{2}P_{ee_0}\cos\theta\,\overline{y}\,\frac{\partial\overline{T}_{i1}}{\partial\overline{y}}+\frac{3}{4}P_{ee_0}\sin\theta\frac{\partial\overline{T}_{i1}}{\partial\theta}=\frac{\partial^{2}\overline{T}_{i1}}{\partial\overline{y}^{2}} \tag{82}$$

The initial and boundary conditions (78) and (79) become:

$$\overline{T}_{e1}(\overline{y},\theta,0)=0, \quad \overline{T}_{e1}(\infty,\theta,\tau)=0,$$

$$\overline{T}_{i1}(\overline{y},\theta,0)=1, \quad \overline{T}_{i1}(-\infty,\theta,\tau)=1 \tag{83}$$

and the interface conditions (80) are :

$$\bar{T}_{e1}(\bar{y},\theta,0)=0\,,\quad \bar{T}_{e1}(\infty,\theta,\tau)=0\,,$$
$$\Omega\frac{\partial\bar{T}_{e1}}{\partial\bar{y}}(0,\theta,\tau)=\frac{\partial\bar{T}_{i1}}{\partial\bar{y}}(0,\theta,\tau)\,. \qquad (84)$$

By analogy to the well-known Chao's solution $[17]$ in the case of a non deforming bubble, we found the solutions of the scaled energy equations (81) and (82) with the conditions (83) and (84) as follows:

$$\bar{T}_{e1}=\frac{T_{e1}-T_\infty}{T_0-T_\infty}=\frac{1}{1+\Omega}erfc\left[\frac{(3P_{ee_0})^{1/2}}{4}\frac{\sin^2\theta}{\varsigma^{1/2}}\gamma^{-1/2}\frac{y}{R_0}\right],\quad y\geq 0\,, \qquad (85)$$

$$\bar{T}_{i1}=\frac{T_{i1}-T_\infty}{T_0-T_\infty}=1-\frac{\Omega}{1+\Omega}erfc\left[\frac{(3P_{ee_0})^{1/2}}{4}(\frac{a_e}{a_i})^{1/2}\frac{\sin^2\theta}{\varsigma^{1/2}}\gamma^{-1/2}\frac{|y|}{R_0}\right],\quad y\leq 0\,. \qquad (86)$$

3.3. Computation of the temperature profiles.

In this section, we apply the formalism detailed above to the case of a deforming gas bubble whose radius grows uniformly with time according to $R(t)=R_0+At$, and which is rising in a surrounding liquid at a constant velocity U_∞. Thus we obtain $\gamma(t)=1+\mathsf{A}\tau_0$, with $\mathsf{A}=A/U_\infty$ and $\tau_0=U_\infty t/R_0$. Computations of the temperature profiles versus $\tilde{y}=y/R_0$ were performed in the plane $\theta=150°$, for three values of $\mathsf{A}\equiv U$, namely 0 (constant radius), 0.1 and 0.2, and for $\tau_0=1$ and $P_{ee_0}=500$, $a_e/a_i=2$, $\Omega=2$.

As a main result, it follows (Fig. 2) that the time scaling of $\tilde{y}$ through $\bar{y}=\gamma^{-1/2}\tilde{y}$ has the great interest. Although the dynamic Reynolds numbers $R_{ee}=2RU_\infty/\nu_e$ and $R_{ei}=2RU_\infty/\nu_i$ increase with time like R(t), the arguments of the erfc-functions in the expressions (85) and (86) decrease with time like $1/\sqrt{R(t)}$. It clearly features the importance of the time scaling laws detailed above on the behavior of the viscid hydrodynamic flows and heat transfers past a deforming spherical bubble, both with time and space.

4 CONCLUSION

As we studied recently $[22]$ the effect of the viscosity on the transient heat

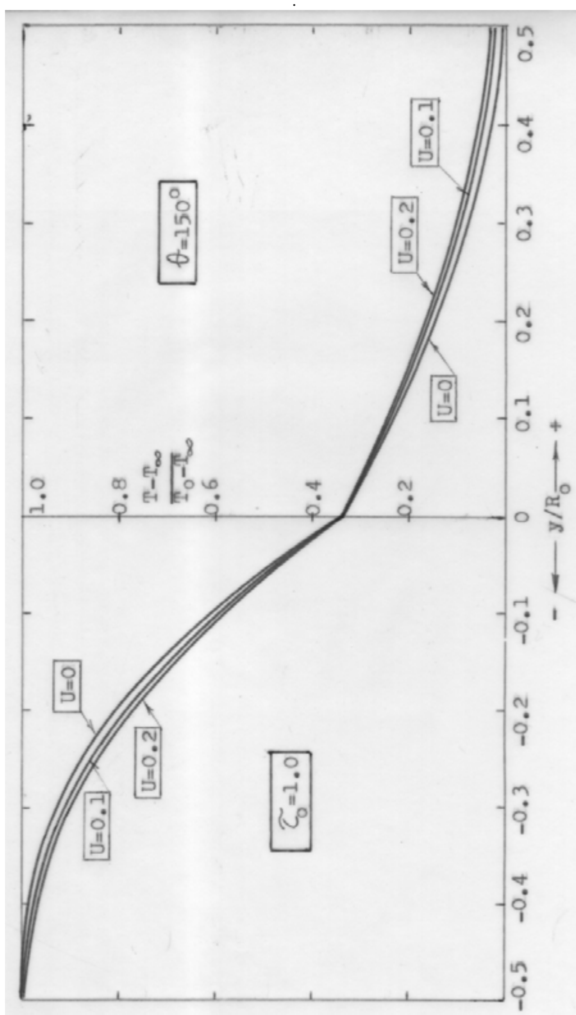

Figure 2. . Transient radial temperature distributions ($P_{ee_0} = 500$, $\Omega = 2$).

transfer to a translating non deformable droplet, the aim of this work is to calculate the viscid contributions to the hydrodynamic flow fields and heat transfers past a deforming spherical gas bubble which is impulsively started in a rectilinear motion at a constant velocity in a viscous fluid of large extent at rest. Exact solutions of the Navier-Stokes and energy equations for the outer and inner flows are obtained within the frame of the thin boundary layer approximation, valid at large Reynolds and Peclet numbers. As the bubble is of a time-dependent radius, it follows that the usual dimensionless parameters, such as Reynolds numbers for instance, are actually dynamic quantities. In order to remove this time-dependence, the equations of motion and heat transfer are rewritten in a appropriate scaled form through powers of a dimensionless quantity defined by the radius of the growing bubble divided by its initial value. In the limit where the rate of deformation is assumed to be small compared to the translation velocity, we show that these scaled equations of motion and heat transfer are analogous to those derived in the static case, provided an appropriate dimensionless time variable is introduced. The solutions for the flow fields and heat transfers derived in this work are valid whatever the law of deformation of the radius of the bubble with time, provided the above-mentioned limit is ensured. A numerical application to the case of a uniformly deforming gas bubble shows clearly the importance of these scaling laws on the space and time behaviors of the hydrodynamic flows and heat transfers

5 PRSPECTIVES

All the exact solutions for the unsteady outer and inner flow fields and heat transfers within the corresponding boundary layers, obtained here, are valid whatever the law of deformation of the bubble radius with time, provided the condition $A \ll 1$ applies. However, in the opposite case, an iterative method of resolution based on double expansions of the velocity components and temperature fields in successive powers of A with coefficients functions of Legendre polynoms can be worked out.

Bubble growth in superheated fluids is of key interest in many industrial problems, such as boiling phenomena in general and in flash evaporation in particular. Most of the large amount of the research on such bubble growth has been conducted first for pure liquids, but also more recently about bubble growth in superheated solutions with a non-volatile solute, a topic of both fundamental and practical importance, with applications including a wide variety of separation processes (such as water desalination) and energy conversion processes (such as ocean-thermal energy conversion), or nuclear reactor safety. Mikic et al. $\left[20\right]$ have developed a simple general equation for calculating bubble growth rates (without motion) in pure liquids, starting with a bubble radius of zero, in the inertia- and heat-transfer-controlled regimes.

$$R^* = \frac{2}{3}\left[(t^*+1)^{3/2} - (t^*)^{3/2} - 1\right] \tag{87}$$

Then Miyatake et al. $\left[21\right]$ have improuved this formula and found a new universal equation for bubble growth (without motion) not only in pure liquids but also in binary solutions with a non-volatile solute, which is valid through all the bubble growth history, i.e., in the surface-tension, inertia-, and heat-transfer-controlled regimes. It would be interesting to study the effect of the viscosity on the flow fields and heat transfers past the interface of a spherical deforming bubble with the time-dependent dimensionless radius (87) for instance rising in a viscous liquid.

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VISKOZNI DOPRINOS STRUJANJU I PRENOSU TOPLOTE PRI KRETANJU SFERNOG DEFORMABILNOG GASNOG MEHURA

Ovoga puta se analiziraju oba granicna sloja, dinamicki i temperaturski, pri naglo pokrenutom DEFORMABILNOM sfernom gasnom mehuru kroz neku viskoznu tecnost, a za velike vrednosti brojeva Rejnolds-a i Pekle-a. Primenom postupka « podesavanja razmera » (scaling procedure) na sve promenljive velicine (prostorne i vremenske koordinate, brzine, temperature) u Navije-Stoksovim i energijskim jednacinama, nadjena su tacna resenja granicnih slojeva s obe strane opne mehura, spoljasnje i unutrasnje. Dobijena teorijska resenja vaze za sporo rastuci mehur, pod uslovima : da ostaje sfernog oblika, da postoji interna cirkulacija i da nema odvajanja spoljasnjeg granicnog sloja

NOMENCLATURE

R_e Reynolds number

P_e Peclet number

R bubble radius

U_∞ bubble velocity

λ thermal conductivity

a thermal diffusivity

μ dynamic viscosity

ν kinematic viscosity

ρ density

e subscript for the external liquid

i subscript for the internal gas bubble

□ $<<$

Sent: Wednesday, May 10, 2006, 9:10 AM

Radomir V. Ašković

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