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THE THEORY OF DIFFERENTIAL EQUATIONS WHICH ARISE IN DYNAMICS OF A SYSTEMS OF RIGID BODIES WITH COULOMB FRICTION

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Summary. *The theory of right solutions of dynamics equations for mechanical systems with sliding friction in one-degree-of-freedom kinematics pairs, which has been developed by the authors, is considered. Some difficulties bound up with "non-uniqueness" of motion in course of description of such systems, which are known as P. Painlevé's paradoxes, are discussed. Some causes of occurrence and possible ways of overcoming these difficulties revealed on the basis of analysis of equations of motion are indicated.*

Keywords: *Coulomb's laws, dry friction, P.Painlevé's paradoxes, equations of dynamics of mechanical systems with friction, right solution, stability.*

1 INTRODUCTION AND PROBLEM STATEMENT

1.1. History of the issue.

For the first time the problem on the possibility of development of a general theory for the systems with friction, which would be similar to the theory based on Lagrange equations as regards to the systems without friction, was stated by P. Painlevé in his publication "Lectures on Friction" [1]. In this work P. Painlevé introduced a general definition of friction forces for the systems of absolutely rigid bodies, and the issue of consistency of links was considered. When describing general properties of friction laws, P. Painlevé indicated to the fact that these are rather rough empirical laws and may be applied only within some definite boundaries, while for large values of friction coefficients application of Coulomb's laws leads to uncertainties. These phenomena known as "P. Painlevé's paradoxes" for dry friction provoked a discussion, in which many outstanding mechanicians of that time participated: L.Lecornue, De Sparre, F. Klein, R. von Mises, G. Hamel, L. Prandtl, F.Pfeiffer (see [1]), E.A.Bolotov [2] (see also [3]), later – N.V. Butenin [4], N.A. Fufayev [5], Yu.I. Neimark, [6],[7], [8], A.P. Ivanov [0], V.V. Nickolsky, Yu.P. Smirnov [11], [12], [13], S.S. Grigoryan [14], S.V. Belokobylsky [15] et al. A detailed survey of the respective publications can be found in the monograph by Le Suan Ane [16]. But presently the works devoted to analysis of P. Painlevé's paradoxes are of discussion character.

Note, paradoxes revealed by P. Painlevé are explicated not in the physical nature of friction but in techniques of its description by methods of theoretical mechanics. These outstand in the possible contradictory character of equations of motion of mechanical systems with friction on the basis of both the assumption of absolute rigidity of contacting (interacting with friction) bodies and the Coulomb's law. P. Painlevé himself has made a conclusion that "there is a logical contradiction between the rigid body dynamics and the Coulomb's laws, which is realized under the conditions which may be realized in reality" [1]. Nevertheless, application of Coulomb's laws under the assumption of absolute rigidity of bodies interacting with friction is considered as justified and has been efficient in many cases in practice.

Friction is a complex physical phenomenon, which is still insufficiently investigated. Laws of friction were studied by Leonardo da Vinci who discovered that when a body moves along a horizontal surface, it experiences the force which obstructs the motion and depends on the body's weight. The same conclusions were later made by G. Amonton. He supposed that the force of friction does not depend on the velocity of relative motion of the bodies interacting with friction. S.O. Coulomb introduced the concept of friction coefficient and concluded that its value is dependent on the material and on the state of surfaces interacting with friction, but does not depend on the surface of contact. The formula $F = fN$ is known as the Admonton-Coulomb law.

Coulomb investigated the force of friction in cases of very slow reciprocal motion of bodies interacting with friction. But already in the XIX century it was found out that the force of friction depends on the relative velocity of contacting bodies. It was discovered: the force of static friction (at the moment of beginning of motion) differs from the force of friction in the process of motion (larger) in the zone of very low velocities. Friction coefficients are dependent not only on the materials, but also on smoothness of the surfaces, which always have some irregularities. So, in reality, the surface of contact of the bodies interacting with friction is relatively small. Resistance of these contact zones is right the source of friction force. In the case of relative shift, there takes place not only sliding but also elastic (Hookean) deformation of microscopic bulges on contacting bodies. In the case of very low shifts, a substantial role belongs to elastic resistance and to the force which shall be subject to Hooke's law. All these and other peculiar properties of friction only add to the conclusion made by P. Painlevé that Coulomb's laws may be applied within definite boundaries and under definite conditions.

Before giving a brief characteristic of the main directions of investigations of systems with friction consider an example of P. Painlevé. This is not an exceptional but a rather general case when the friction coefficient values are large. Such an example has been analyzed in detail in [17].

Consider two material points of unit mass, which are linked by a weightless rod MM_1 of length $r > 0$. Some point M is sliding with friction along an immovable horizontal straight axis Ox, which it cannot leave, and has a coordinate x, and another point M_1 is moving without any external resistance in a vertical plane Oxy under the effect of the gravity force g (and reaction of the rod). Axis Oy is directed downwards, θ is the angle of deviation of the rod from the positive direction of the axis Ox clockwise. The coefficient of friction $f > 0$ is considered to be positive. The external forces acting upon the system are the complete weight $2g$, applied to the center of gravity G, and the reaction R of the axis Ox, whose components along the axes Ox and Oy will be denoted as R_x and R_y. The equations of motion and the equations for the normal reaction R_y write:

$$\begin{aligned} &2\ddot{x} - r\sin\theta\,\ddot{\theta} = r\dot{\theta}^2\cos\theta + R_x \\ &-\sin\theta\,\ddot{x} + r\ddot{\theta} = g\cos\theta \\ &r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta - 2g = R_y \end{aligned} \tag{1}$$

The absolute value of the tangential reaction R_x for $\dot{x} \neq 0$ (the force of sliding friction in motion) is $f\left|R_y\right|$ and has the sign opposite to that of velocity $\dot{x}$ of point M, where $f > 0$ is the friction coefficient (a constant value). Therefore, according to Coulomb's law, in case of motion with $\dot{x} \neq 0$ we have

$$R_y = -f\left|R_y\right| sgn\dot{x} \tag{2}$$

Let $0 < \theta_0 < \pi/2$. Having chosen the sign of velocity $\dot{x}_0$ and the direction of the force of normal reaction R_y (i.e. getting rid of the sign of $\left|R_y\right|$), we obtain a system of linear equations with respect to the reaction R_y and the accelarations $\ddot{\theta}$ and $\ddot{x}$. Analysis of these equations shows that under definite conditions applied to the friction coefficient f these have no solutions for $\dot{x}_0 > 0$ (the paradox of impossibility of motion), and for $\dot{x} < 0$ there are two possible cases satisfied, which do not contradict to the law of friction (2) (the paradox of nonuniqueness).

Already in course of above discussion they formed a direction of investigations of systems with friction, which presumed introduction of physics hypotheses which add to Coulomb's laws.

The idea of taking account of elasticity of real bodies was put forward by L. Lecornue after initiation of the discussion. He was sure that the force of friction appeared not immediately and the friction coefficient grew from zero to the value, which corresponded to the Coulomb's law value, during a very short time interval, i.e. he took account of the tangential elasticity of interacting bodies. Nowadays the L. Lecornue approach has been developed in [15].

F. Klein and L. Prandtl tried to circumvent the contradictions by assuming the admissibility of infinite accelerations, i.e. at the expense of instantaneous change of the velocity (within the frames of the hypothesis of absolutely rigid body). Actually, this may imply a very fast halting of the body, which, in particular, expresses the hypothesis of instantaneous braking. As the discussion developed, L. Prandtl and F. Pfeiffer proposed also to take account of the normal elasticity of bodies in the contact zone. The investigation was conducted on the system which included two material points linked by an elastic rod and moving along parallel directions. One of the points experienced friction. This example was completely investigated, and no paradoxes arised. Now there appeared the possibility to investigate the behavior of the system's solution side by side with the increase of elasticity (to perform the passage to the limit). The papers [4] and [18] realized this idea. But it is difficult to investigate limit equations in in the general form (without any idea of the solution in its analytical form). These difficulties are still unresolved. Moreover, in the general case, it has not been strictly grounded that the P.Painlevé contradictions can be resolvede on account of elastic (Hookean) deformations in the contact zone.

The account of elasticity leads to the necessity of considering differential equations with a small parameter with the elder derivatives. The theory of A.N. Tikhonov was applied in [5], where the value inverse to the elasticity coefficient was considered in the capacity of the small parameter.

In [14], an approach based on replacement of the Coulomb's law with another "physically real law with a distributed value of the friction force, which is achievable under large values of the normal force on the surface of friction" was proposed. That was grounded by the fact that in rigid body mechanics and in mechanics of deformable bodies normal pressures may be arbitrary large, while the tangential impacts (forces of friction) on the surface of contact of the bodies may be restricted by he factor of their stength.

A hypothesis of tangential impact, which added to the Coulomb's law (instantaneous selfbraking) was investigated in [9]. .

Other investigations in this direction are also known. These are related to investigation and explanation of P. Painlevé's paradoxes, which are considered via definite examples of systems with friction.

The publications [11], [12], [13] develop the method of rejection of irrelevant solutions and choosing the "true" motion of possible ones at the expense of consideration of systems having bilateral holding links. The authors have proposed the principle of invariance of the links for the systems of absolutely rigid bodies with vanishing small clearances in the holding bilateral links. Different variants relace one another in the process of motion. It is necessary to choose the noncontradictory motion from possible ones, which would be unique with respect to the signs of normal reactions – i.e. the "true motion". The paradox of non-uniqueness of motion here results from multi-variant character of contacts between bodies. The paradox of impossibility of motion results from the absence of a non-contradictory variant of links. These ideas go back to investigations of P. Painlevé. But there arise substantial difficulties in the process of investigation of friction for rotating motions, because there appears some quadratic dependence of the friction force on normal reactions.

The discussion, which was initiated over a hundred years ago, is under way. Mechanicians of different schools and research directions participate in it. But still there are results explaining Penleve's paradoxes, which might be recognized as canonic ones. It is still unknown in what degree different investigations correlate or complement one another. The comparative analysis is practically impossible.

For example, in [8] it is affirmed that there is no limit dynamics, which corresponds to the transition to an absolutely rigid body when the regulation parameters tend to zero. The conclusion is made that it is expedient to relinquish the hypothesis and the problem of choosing “true motions in paradoxical situations” in mathematical models of systems with friction. Instead, it is necessary to solve the problem of “model’s completeness”, which implies the need to construct a minimum sufficient correct model in a paradoxical situation.

Such an attention to friction paradoxes is mainly bound up with the tendency of researchers to construct a complete and non-contradictory theory of systems with friction. But the object of investigations is one of the most complex ones concerning its nature, and probably the point is hardly ever expected in future.

Meanwhile, investigations of systems with friction within the frames of analytical mechanics of systems with friction are not reduced to analysis of contradictions. There are fundamental works of P. Appel [19], [17] , A.I. Lurye [[19] N.G. Chetayev [21], V.V. Rumyantsev [22], G.V. Pozharitsky [23], [24], [25], in which the Euler-Lagrange principle of possible transitions, the method of Lagrange and the Gauss principle of the least compulsion are applied (extended) to the systems with friction. The publications [26]], [27], [27], [29], in which the theory of of systems with dry friction is developed (without any account of the possibility of occurrence of P. Painlevé’s paradoxes), are to be noted. Note also rather interesting and pithy surveys related to friction [31], [32].

In conclusion of this section we have to note another peculiarity in description of systems with friction: the force of friction is a discontinuous function of velocity, and so, motion of systems with Coulomb’s friction is described by differential equations with discontinuous right-hand sides, the theory of which is presently well developed (see [33]). This circumstance adds substantial but purely mathematical difficulties related to investigation of equations of motion for the systems with friction. It is not related to P. Painlevé’s paradoxes: P. Painlevé’s paradoxes are revealed in the domains of continuity of right-hand sides of the equations of motion. Note also, differential equations with the discontinuous right-hand side were for the first time considered right in the dynamics of mechanical systems with Coulomb’s friction in the works of P. Painlevé and P. Appel.

1.2. Some general remarks of sliding friction.

Consider some known concepts and information related to systems with sliding friction (see [17: P.107]), which will be used in description of friction forces.
Imagine two moving rigid bodies A and B, body A slides along body B, m is the point of touch. N is the normal reaction of body B onto body A, i.e. the force directed perpendicularly to the surfaces touching at point m, F is the force applied to the same point m and acting in the plane tangential with respect to the touching surfaces. Force F is said to be the force of sliding friction. It is acts in the direction opposite to the relative velocity of point m with respect to B and is equal to fN, where f is the friction coefficient, and N is an absolute value of the normal reaction. Therefore, the force of sliding friction can be determined in the process of motion, when values of f and N are known. The friction coefficient f is determined experimental and is dependent on the nature of touching surfaces. Note, the friction coefficient at rest is somewhat smaller than that in case of motion. In this case, following [1], we consider these coefficients to be equal.

What happens if the relative velocity of point m with respect to body B turns zero, i.e. the sliding motion terminates? In this case, either body A remains immovable with respect to body B, or the relative motion is represented by the rolling motion, and the laws of sliding friction n motion may not be applied. Having neglected rolling and turming kinds of friction, let us formulate a general principle of defining the force of sliding friction in the state of relative rest, i.e. at the moments of termination of sliding motion.

Suppose, the relative velocity of point B at the initial time moment t_0 is zero. It is necessary to find out what kind of relative motion of body A along body B will be realized at the following time moments $t > t_0$. In order to answer this question let us behave as follows: assume that when $t > t_0$ the velocity of point m remains zero. Hence, on account of above laws, the reaction of body A on body B will be comprised by the normal reaction N and the tangential reaction F. It is assumed that in this case laws of friction in the state of rest may be applied, and so, the inequality $F < fN$ shall hold. Under these conditions let us formulate equations of the problem and compute the values of N and F. If indeed the value of F obtained turns out to be smaller than fN, then the assumption is plausible, i.e.: the relative velocity of point m at $t > t_0$ is zero, and F is the force of sliding friction at rest. This statement is plausible until the value of F becomes larger than fN. Beginning from this moment, there will take place the process of sliding, and the equations will have to be changed. If, vice versa the value of F obtained will from the very beginning be larger than fN, then the assumption (that the velocity of point m at $t > t_0$ is zero) made before is false, and sliding motion is hardly ever possible. From the very beginning it is necessary to write equations of motion, while applying laws of sliding friction not at rest, but for the case of motion.

In his case, when sliding friction takes place both before the moment of stopping t_0 and after this moment, according to the law of friction in motion, after the change of the direction of motion to the reverse of one, the friction will change the sign to the opposite one, i.e. a stepwise change of the friction force (noted above) will happen.

Above reasoning is rather schematic and general. In concrete situations, it is insufficient to derive only equations of motion in order to determine the friction force. If active forces, which influence the system, are known, then the reactions of the links have to be computed and analyzed. When there are many points of friction interaction of bodies all above considerations have to be applied with respect to each of the points. The method of determination of the reactions of friction interactions is considered in the next section.

1.3. General equations of motion of systems with one-degree-of-freedom kinematic pairs with friction.

Our investigations are conducted within the frames of classical rigid body mechanics.
Considered are Lagrange II-kind equations, which describe motion of systems with

Coulomb's sliding friction, with ideal (perfect) nonstationary holding links. As noted above, according to Coulomb's laws, generalized forces of sliding friction may be expressed via friction coefficients and modules of normal reactions at the points of contact between two bodies interacting with friction. The latter are written via the generalized reactions of the friction links, which are not known in advance. In the present paper we propose and approach, which is based on consideration of Lagrange equations with multipliers and excessive generalized coordinates и избыточными обобщенными координатами and under the condition of conceptual liberation of the initial system from the links, which cause the reactions to be found (see [19]). As a result, we obtain an "extended" system of equations, which – in the capacity of unknown functions – includes not only the system's solution but also generalized reactions of the friction interactions. Since the modules of normal reactions are nonlinear functions, the generalized reactions of the friction interactions are given by such equations inexplicitly. Solution of such systems "extended" with respect to systems' generalized accelerations is not always possible and not always unequivocal. This is just what P. Painlevé's paradoxes are explicated in, and their investigation acquires a purely mathematical character.

Let us consider the approach discussed above in detail. Let there be given a mechanical system with k degrees of freedom, in which Coulomb friction forces act on $k_* \le k$ kinematic pairs described by the generalized coordinates $q^1, \dots, q^{k_*}$. For the purpose of liberation of the system from interactions bound up with friction, let us introduce $k^* \ge k_*$ additional generalized coordinates $q_*^1, \dots, q_*^{k^*}$, which are chosen such that equations of "broken" links (interactions) would have the most simple form:

$$q_*^j = 0, \quad j = 1, \dots, k^* \tag{1}$$

The residual links (interactions) without friction have been retained and are assumed to be perfect, holonome, and, generally speaking, nonstationary.

Introduce the denotations: $q = \left(q^1, \dots, q^k\right)$, $q_* = \left(q_*^1, \dots, q_*^{k^*}\right)$ are vectors of generalized coordinates of the initial system and additional generalized coordinates; $N' = \left(N_1', \dots, N_{k^*}'\right)$ is the vector of generalized reaction forces of "broken" links (interconnections). Such signs as "one dot" and "two dots" on top of the symbols of generalized coordinates denote, respectively, the first and the second derivatives with respect to time.

The kinetic energy T_{a^*} and the generalized forces Q_i^* are composed for the system liberated from links for the joined vector of generalized coordinates $q_\sigma = \left(q_\sigma^1, \dots, q_\sigma^{k+k^*}\right)$ (i.e. $q_\sigma^i = q^i$ $(i = 1, \dots, k)$, $q_\sigma^i = q_*^{i-k}$ $(i = k+1, \dots, k+k^*)$). According to the assumption, T_{a^*} represents a sum of positive definite form of generalized velocities, of the linear form of generalized velocities $\dot{q}_\sigma^i$ and of function $T_0^* = T_0^*\left(t, q_\sigma\right)$:

$$T_{a^*} = \frac{1}{2}\sum_{i=1}^{k+k^*}\sum_{j=1}^{k+k^*} a_{ij}^*\left(t,\, q_\sigma\right)\dot{q}_\sigma^i \dot{q}_\sigma^j + \sum_{i=1}^{k+k^*} a_i^*\left(t,\, q_\sigma\right)\dot{q}_\sigma^i + T_0^*\left(t, q_\sigma\right).$$

The functions $a_{ij}^*\left(t,\, q_\sigma\right)$, $a_i^*\left(t,\, q_\sigma\right)$, $T_0^*\left(t,\, q_\sigma\right)$ are assumed continuously differentiable with respect to the set of its arguments.

Equations of motion for the "extended" system in Lagrange form may be written as follows:

$$\frac{d}{dt}\frac{\partial T_{a^*}}{\partial \dot{q}^i} - \frac{\partial T_{a^*}}{\partial q^i} = Q_i^*, \quad i = 1, \ldots, k \tag{2}$$

$$\frac{d}{dt}\frac{\partial T_{a^*}}{\partial \dot{q}_*^j} - \frac{\partial T_{a^*}}{\partial q_*^j} = Q_{k+j}^* + N_j', \quad j = 1, \ldots, k^* \tag{3}$$

The generalized forces Q_i^* have the form $Q_i^* = Q_i^{A^*}\left(t,\, q_\sigma,\, \dot{q}_\sigma\right) + Q_i^{T^*}\left(t,\, q_\sigma,\, \dot{q}_\sigma,\, N'\right)$, $i = 1, \ldots, k + k^*$, where $Q_i^{A^*}$ are generalized active forces acting on the system (potential forces; forces of resitance of the dampers, the environment; disturbing and controlling forces of any physical nature, forces of radial correction); $Q_i^{T^*}\left(t,\, q_\sigma,\, \dot{q}_\sigma,\, N'\right)$ are generalized friction forces. It is supposed that the work of friction forces on the virtual transitions along additional generalized coordinates is zero.

When considering equations (2), (3) consistent with the equations of interconnections (links) (1) and substituting the values of $q_*^j = 0$, $\dot{q}_*^j = 0$, $\ddot{q}_*^j = 0$, $j = 1, \ldots, k^*$ into them (due to equations of links), we obtain $k + k^*$ equations needed for defining the generalized coordinates $q^1, \ldots, q^k$ and the generalized reactions of interconnections with friction $N_1', \ldots, N_{k^*}'$ (in the domain related to determination of frictions forces). In the expanded for these equations write as follows:

$$\sum_{s=1}^{k} a_{i,s}\left(t,\, q\right)\ddot{q}^s = g_i\left(t,\, q,\, \dot{q}\right) + Q_i^A\left(t,\, q,\, \dot{q}\right) + Q_i^T\left(t,\, q,\, \dot{q},\, N'\right), \quad i = 1, \ldots, k \tag{4}$$

$$\sum_{s=1}^{k} a_{k+j,s}\left(t,\, q\right)\ddot{q}^s = g_{k+j}\left(t,\, q,\, \dot{q}\right) + Q_{k+j}^A\left(t,\, q,\, \dot{q}\right) + N_j', \quad j = 1, \ldots, k^* \tag{5}$$

where $a_{l,s}\left(t,\, q\right) = a_{l,s}^*\left(t,\, q_\sigma\right)\Big|_{q_*=0, \dot{q}_*=0}$, $\left(l = 1, \ldots, k + k^*,\ s = 1, \ldots, k\right)$ are are the coefficients of the quadratic form of generalized velocities $\dot{q}_\sigma^l$ from the expression of the kinetic energy T_{a^*} under the condition that $q_* = 0$; $Q_l^A\left(t,\, q,\, \dot{q}\right) = Q^{A^*}\left(t,\, q_\sigma,\, \dot{q}_\sigma\right)\Big|_{q_*=0,\, \dot{q}_*=0}$, $\left(l = 1, \ldots, k + k^*\right)$ are generalized active forces; $g_l\left(t,\, q,\, \dot{q}\right) = g_l^*\left(t,\, q_\sigma,\, \dot{q}_\sigma\right)\Big|_{q_*=0,\, \dot{q}_*=0}$, $\left(l = 1, \ldots, k + k^*\right)$ are continuous functions, which characterize generalized gyroscopic forces, and some other terms under the condition that $q_* = 0$, $\dot{q}_* = 0$.

The system (4), (5) represents a specific system of 2nd order differential equations with the discontinuous right-hand side, in which the generalized reactions of interconnections (links) $N_i'(t)$ and the system's motion $q^i(t)$ are unknown functions. This system may be called algebraic-differentiable, except for the only difference that functions $N_i'(t)$ aare included in it without their derivatives.

Let us use the group of equations (7) to express reactions of interconnections (links) via the generalized accelerations and substitute them into equations (6). As a result, we obtain the equations of motion in their vector and inexplicit form:

$$A(t, q)\ddot{q} = g(t, q, \dot{q}) + Q^A(t, q, \dot{q}) + Q^T(t, q, \dot{q}, \ddot{q}) \qquad (6)$$

Friction forces in this system are dependent only on the velocities. And the question may arise: whether this approach takes the problem beyond the frames of Newtonean mechanics, in which any dependence of forces on accelerations is inadmissible, or not. In this connection note that equations (6) represent just a form recording the initial "extended" equations of motion in inexplicit form. Note also, the forms of recording equations of motion for the systems with friction represent another special objective of investigations.

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1.4. Generalized forces of sliding friction.

Let the interconnections (links) and the generalized coordinates $q^1, \ldots, q^k$ are such that each of the generalized friction forces Q_i^T depends explicitly only on one respective generalized velocity $\dot{q}^i$ and on normal reactions N_i' . The latter can be expressed from equations (5) as functions of time, generalized states, velocities and accelerations. According to Coulomb's laws, in the process of motion with generalized velocities $\dot{q}^s(t) \neq 0$, $(s = 1, \ldots, k_*,\ 1 \le k_* \le k)$, generalized sliding friction firces Q_i^T are expressed by the formulas $Q_s^{T1} = -f_s \,|\, N_s \,|\, sgn\, \dot{q}^s$, $(s = 1, \ldots, k_*)$ via friction coefficients (in motion) f_s and modules of normal reactions. The modules of normal reactions are expressed by the equalities: $|\, N_s \,| = |\, N_s' \,|$ in the process of sliding with velocity $\dot{q}^s$ along the surface $q_*^s = 0$, $s = 1, \ldots, k'$, $0 \le k' \le k_*$; $|\, N_s \,| = \sqrt{N_j'^2 + N'_{,j}{}^2}$, when N_s is obtained by adding mutually orthogonal reactions of smooth interconnections (links) $q_*^j = 0$, $q_*^{j'} = 0$ with the absolute values $\left| N'_j \right|$, $\left| N'_{j'} \right|$, $s = k'+1, \ldots, k_*$, $i,\ j = k'+1, \ldots, k^*$ for the case of rotation of the body about the axis $q^j = 0$, $q^{j'} = 0$ or for the case of sliding of a point along a spatial curved line.

Note, functions $\left| N_s \right|$ are continuous with respect to $\ddot{q}$, and for $\left| N_s \right| \neq 0$ – continuously differentiable with respect to $\ddot{q}$.

So, in case of generalized forces of sliding friction in motion we have в движении

$$Q_s^{T1} = -f_s\left(t,\ q^s,\ \dot{q}^s\right)\left|N_s\left(t,\ q,\ \dot{q},\ \ddot{q}\right)\right| sgn\dot{q}^s \text{ если } \dot{q}^s \neq 0,\ \ s = 1, \ldots, k_* \quad (7)$$

Let now the velocity of sliding friction of the body interactiong with onother one with friction at some time moment is zero. According to the laws of classical mechanics, let us suppose that the friction coefficients $f_s^0\left(t,\ q^s\right)$ for the case of relative rest are equal to the coefficients of friction in motion, i.e. $f_s^0\left(t,\ q^s\right) = f_s\left(t,\ q^s,\ 0\right),\ s = 1, \ldots, k_*$.

Next, we act according to the scheme discussed in Section 1.3 for the purpose of determination of friction forces in caseof relative rest. If $\dot{q}^s\left(t\right) = 0$, where the index is $1 \leq s \leq k_*$, then we obtain $\ddot{q}^s = 0$ and can compute the generalized force of sliding friction in case of relative rest

$$Q_s^{T0}\left(t,\ q,\ \dot{q},\ \ddot{q}\right) @ \sum_{j=1, j\neq s}^{k} a_{sj}\left(t,q\right)\ddot{q}^j - \left[g_s\left(t,\ q,\ \dot{q}\right) - Q_s^A\left(t,\ q,\ \dot{q}\right)\right]_{\dot{q}^s=0}.$$

When the equality

$$\left| Q_s^{T0}\left(t,\ q,\ \dot{q},\ \ddot{q}\right) \right| \leq f_s^0\left(t, q^s\right)\left|N_s\left(t,\ q,\ \dot{q}, \ddot{q}\right)\right|_{\dot{q}^s=\ddot{q}^s=0} \quad (8)$$

holds, we naturally have $\ddot{q}^s = 0$ and $Q_s^T\left(t,\ q,\ \dot{q},\ \ddot{q}\right) = Q_s^{T0}\left(t,\ q,\ \dot{q},\ \ddot{q}\right)$.

In case when inequality (8) fails to hold, the assumption made (that $\ddot{q}^s = 0$) is rejected, and it is supposed that

$$Q_s^T\left(t,\ q,\ \dot{q},\ \ddot{q}\right) = f_s\left(t,\ q^s,\ 0\right)\left|N_s\left(t,\ q,\ \dot{q},\ \ddot{q}\right)\right| sgn\, Q_s^{T0}\left(t,\ q,\ \dot{q},\ \ddot{q}\right) \quad (9)$$

Hence, for the case of actual system's motion, inequality $\ddot{q}^s \neq 0$ holds, because for $\ddot{q}^s = 0$ we would have $\left| Q_s^{T0} \right| = f_s^0 \left| N_s \right|_{\dot{q}^s=\ddot{q}^s=0}$

In the general case, we obtain the following expression for the generalized force of sliding friction:

$$Q_s^T\left(t,\, q,\, \dot{q},\, \ddot{q}\right)=\begin{cases}-f\left(t,\, q^s,\, \dot{q}^s\right)\left|\, N_s\left(t,\, q,\, \dot{q},\, \ddot{q}\right)\right| sgn\,\dot{q}^s, \text{ if } \dot{q}^s \neq 0, \\ f_s\left(t, q^s, 0\right)\left|\, N_s\left(t,\, q,\, \dot{q},\, \ddot{q}\right)\right| sgn\,Q_s^{T0}\left(t,\, q,\, \dot{q},\, \ddot{q}\right), \text{ if } \dot{q}^s = 0, \\ \left|\, Q_s^{T0}\left(t,\, q,\, \dot{q},\, \ddot{q}\right)\right| > f_s^0\left(t,\, q^s\right)\left| N_s\left(t,\, q,\, \dot{q},\, \ddot{q}\right)\right|_{\ddot{q}^s=0}, \\ Q_s^{T0}\left(t, q, \dot{q}, \ddot{q}\right), \text{ if } \dot{q}^s = 0, \\ \left|\, Q_s^{T0}\left(t,\, q,\, \dot{q},\, \ddot{q}\right)\right| \leq f_s^0\left(t, q^s\right)\left|\, N_s\left(t,\, q,\, \dot{q},\, \ddot{q}\right)\right|_{\ddot{q}^s=0}\end{cases} \quad (10)$$

So, in the present paper we investigate equation (8) with the forces of sliding friction, which are defined by formula (10).

Hence there appear the two problems:

1) The problem of unequivocal solving of equations of motion (6) with respect to generalized acceleration and reducing them the normal form

$$\ddot{q} = G\left(t,\, q,\, \dot{q}\right) \quad (11)$$

2) The problem of development of the general theory and methods for investigation of equation (11).

In the present paper we elaborate the theory of right solutions of equations (8). The right solution defined on some segment $\left[t_0,\, t_1\right)$ is understood as a continuous right differentiable vector function $\left(q\left(t\right),\, \dot{q}\left(t\right)\right)$, which satisfies the conditions

$$D^+ q\left(t\right) = \dot{q}\left(t\right), \quad D^+ \dot{q}\left(t\right) = G\left(t,\, q\left(t\right),\, \dot{q}\left(t\right)\right)$$
$$q\left(t_0\right) = q_0, \quad \dot{q}\left(t_0\right) = \dot{q}_0 \quad (12)$$

Note, the concept of right solution is to the highest degree corresponding to the sense and to the pithiness of the problem under scrutiny. There may exist no classical solutions in this case because function G is discontinuous. At the same time, the right derivative of the velocity has the sense of generalized acceleration in mechanics.

The difficulty of investigation of the equation implies that function G is not only discontinuous, it is inexplicitly assigned. This complicates application of well-elaborated methods of the theory of discontinuous systems such as the method of transition to differential inclusions. So, for the purpose of investigation of the problems stated above (within the frames of the theory of differential equations with the discontinuous right-hand sides) we have identified and investigated a special class of discontinuous systems. The next Part 2 of the paper is devoted to this issue.

2 RIGHT SOLUTIONS OF DIFFERENTIAL EQUATIONA WITH DISCONTINUOUS RIGHT-HAND SIZES

2.1. Existence and general properties of solutions.

Consider the following differential equation in vector form:

$$\dot{x} = f(t,\ x) \tag{13}$$

which has a discontinuous function $f : \Omega \to R^{n+1}$ unequivocally defined at each point $(t,\ x)$, where $\Omega \subset R^{n+1}$ is a domain.

Analysis of equations (6) has given evidence that there is a set properties of function G, within the frames of which for equation (13) it is possible to consider a sufficiently complete mathematical theory. Generalization of these properties taken in the capacity of propositions reveals in essence a new class of differential equations with the discontinuous right-hand side, for which it is possible to slove the principal set of problems of the general theory (existence, continuability of solutions, their dependence on the initial conditions, etc.) and develop methods of the qualitative theory. Such a generalized problem statement, on the one hand, allows one to demonstrate what properties of scrutinized equations of motion for some systems with friction are the most important in the aspect of their investigation by mathematical methods, and on the other hand, allows one to directly apply results and methods of investigation to the systems of different nature.

The right solution of the initial value problem for the equation (13) on the segment $[t_0,\ t_1)$ with the initial condition $(t_0,\ x_0) \in \Omega$ is understood as an absolutely continuous right differentiable function $x(t)$, which satisfies the conditions $D^+x(t) = f(t,\ x(t))$, $x(t_0) = x_0$ for all $t \in [t_0,\ t_1)$, where $D^+x(t)$ is the right derivative of function $x(t)$. Henceforth, solutions of equation (13) are everywhere understood as right.

For each point $(t,\ x) \in \Omega$ the inclusion $\Gamma = \Gamma(t,\ x) \subset R^n$ denotes a nonempty closed cone (i.e. Γ is a closed set, and for any number $\alpha \geq 0$ from the condition $y \in \Gamma$ it is possible to derive that $\alpha\, y \in \Gamma$). The cases of $\Gamma = R^n$ or $\Gamma = \{0\}$ are not excluded.

Suppose that

$$S_\delta(t,\ x) @ \{(t', x') : t \leq t' < t + \delta,\ \| x - x' \| < \delta\},$$

$$\Omega^0(t,\ x) @ \{(t',\ x') : x' \in x + \Gamma(t,\ x)\},$$

$$\Omega^0_\delta(t,\ x) @ \Omega^0(t,\ x) \cap S_\delta(t,\ x).$$

Suppose also the following conditions are satisfied for the equation (13):

M a i n c o n d i t i o n s . *For each point* $(t,\, x)\in\Omega$ *defined are the set* $\Gamma(t,\, x)$ *and the real-valued continous function* $V_{(t,x)}(t',\, x')$*, which is Lipschitz with respect to* x' *uniformly with respect to* t' *in some neighborhood* $S_\delta(t,\, x)$*, for which (including function* f *) the following conditions hold:*

1. *Function* f *is locally bounded;*
2. *Function* f *is continuous at each point* $(t,\, x)$ *along the set* $\Gamma(t,\, x)$ *right with respect to* t'*, i.e. for any* $\varepsilon>0$ *there exists* $\delta=\delta(\varepsilon,\, t,\, x)>0$ *such that* $\left\| f(t,\, x)-f(t',\, x')\right\|<\varepsilon$ *for all* $(t',\, x')\in\Omega^0_\delta(t,\, x)$*;*
3. $f(t,\, x)\in\Gamma(t,\, x)$ *for all* $(t,\, x)\in\Omega$*;*
4. *The following conditions hold for all* $(t',\, x')\in S_\delta(t,\, x)$:

$$V_{(t,x)}(t',x')\geq 0 \text{ and } V_{(t,x)}(t',x')=0 \Leftrightarrow (t',x')\in\Omega^0_\delta(t,\, x),.$$

and if $\Gamma(t,\, x)\neq R^n$*, then there exist numbers* $\alpha_0=\alpha_0(t,\, x)>0$ $\delta=\delta(t,\, x)>0$ *such that* $D^{*+}V_{(t,x)}(t',\, x')<-\alpha$ *for all* $(t',\, x')\in S_\delta(t,\, x)\setminus\Omega^0(t,x)$*, where*

$$D^{*+}V_{(t,x)}(t',\, x') @ \overline{\lim_{h\to+0}}\, \frac{V_{(t,x)}\big(t'+h,\, x'+hf(t',\, x')\big)-V_{(t,x)}(t',\, x')}{h}.$$

Note that at each point $(t,\, x)\in\Omega$ such that $\Gamma(t,\, x)=R^n$ conditions 1–4 hold if (it is sufficient) function f is continuous at this point. Note also, function $V_{(t,x)}(\cdot,\cdot)$ (for the fixed $(t,\, x)$), which is essentially Lyapunov function, in the beginning serves for proving the existence of solutions and only later the expression $D^{*+}V_{(t,x)}(t',\, x')$ is considered as the upper right derivative of the function $(t',\, x')\to V_{(t,x)}(t',\, x')$ along the solution of equation (13) and is used for investigation of stability. Function $V_{(t,x)}$ with assigned properties exists and is constructed below for the systems systems with friction under scrutiny.

Let us formulate the principal theorems on existence and on properties of solutions of equation (13), whle assumeing that the principal conditions are satisfied.

T h e o r e m 1.1.1. *For any initial state* $(t_0,\, x_0)\in\Omega$ *there exists a local right solution of equation (*13*).*

D e f i n i t i o n . *The right solution* $x(t)$ *of equation (*13*) is called* R *– right* *when function* $D^+x(t)$ *(right derivative of solution* $x(t)$*) is right continuous at each point* t *from the domain of definition for* $x(t)$.

T h e o r e m 1.1.2. *Any right solution of equation (*13*) is* R *– right.*
The concepts of continuability of the solution, noncontinualble solution, right maximal interval of existence are understood in the general sense.

T h e o r e m 1.1.3. *Any solution of equation (*13*) may be continued onto the right maximal interval of existence* $[t_0, \omega)$ *by the initial conditons* $(t_0, x_0) \in \Omega$. *Any right noncontinualble solution of equation (*13*) tends to the boundary of set* Ω *(i.e. leaves any compact subset from* Ω*).*

Let Λ be some metric space with metric $d(\cdot,\cdot)$, and function f be dependent also on the variable $\lambda \in \Lambda$ considered as the parameter. Consider the equation

$$\dot{x} = f(t, x, \lambda) \tag{14}$$

with the function $f : \Omega \times \Lambda \to R^n$ and the initial data (t_0, x_0) under the condition that $\lambda = \lambda_0$. Assume that for any fixed $\lambda \in \Lambda$ conditions 1–4 formulated above hold for the function $f(\cdot, \cdot, \lambda)$. As far as dependence of function f on parameter λ is concerned, suppose the following: for each $\lambda_0 \in \Lambda$, for any compact set $W \subset \Omega$ and $\varepsilon > 0$ there exists $\delta = \delta(\lambda_0, \varepsilon, W) > 0$ such that the inequality $\| f(t, x, \lambda) - f(t, x, \lambda_0) \| < \varepsilon$ holds for all $(t, x) \in W$ and for the values of $\lambda \in \Lambda$, which satisfy the condition $d(\lambda, \lambda_0) < \delta$.

The property of right uniqueness of equation (16) means that side by side with the increase of t for the fixed λ its solutions may merge but cannot branch.

T h e o r e m 1.2.1. *Let equation (*14*) possess the property of right uniqueness,* $x(t)$ *be the solution of problem (*14*) with the initial data* (t_0, x_0)*, with the value of* $\lambda = \lambda_0$ *and with the right maximal interval of existence* $[t_0, \omega)$. *Hence for any* $\varepsilon > 0$ *and* $t^* \in (t_0, \omega)$ *there extsts* $\delta > 0$ *such that any right solution* $x'(t)$ *of problem (*14*) with the initial states* (t_0', x_0') *and with the values of* λ'*, which satisfy the conditions* $\| x_0' - x_0 \| < \delta$, $t_0 - \delta < t_0' \le t_0$, $d(\lambda', \lambda_0) < \delta$*, may be continued onto the right maximal interval of existence* $[t_0', \omega')$, $\omega' > t^*$*, and (for the continuation of* $x'(t)$*) the inequality* $\| x'(t) - x(t) \| < \varepsilon$ *holds for all* $t \in [t_0, t^*]$.

Consider some properties of the intehgral funnel of equation (14) without any assumption of right uniqueness. Let the set $\Omega(t_0) @ \{x : (t_0, x) \in \Omega\}$ be nonempty and $A \subset \Omega(t_0) \times \Lambda$. By $H_f(A)$ we denote the set of all noncontinualble solutions $x(t)$ of equation (14) with the initial data $(x(t_0), \lambda_0) \in A$. Under the assumption that all the solutions $x(\cdot) \in H_f(A)$ are defined on the segment $[t_0, a)$ and $t^* < a$, by $H_f(A)[t_0, t^*]$ we denote the set of all narrowings of such solutions onto the segment $[t_0, t^*]$.

The set $\Phi_f(A)[t_0, t^*] @ \{(t, x(t)) : x(\cdot) \in H_f(A),\ t_0 \le t \le t^*\}$ is called a segment of the integral funnel of equation (14), which lies within the band $t_0 \le t \le t^*$.

T h e o r e m 1.3.1. *let the set $A \subset \Omega(t_0) \times \Lambda$ be compact. Hence the set of solutions $H_f(A)[t_0, t^*]$ is compact in the space $C([t_0, t^*])$ of continuous functions defined on the segment $[t_0, t^*]$ with the topology of uniform convergence. The segment of the integral funnel $\Phi_f(A)[t_0, t^*]$ is a compact set of space R^{n+1}.*

C o r o l l a r y 1.3.1. *Let $(t_0, x_0) \in \Omega$, $\lambda_0 \in \Lambda$ be fixed, and all the solutions of (*14*) with the initial conditions (t_0, x_0) for $\lambda = \lambda_0$ be defined on the segment $[t_0, a)$. Hence for any $t^* \in (t_0, a)$ and for some sufficiently small $\delta > 0$ for all x_0', λ', which satisfy the condition*

$$(t_0, x_0') \in \Omega, \ \| x_0' - x_0 \| < \delta, \ \ d(\lambda', \lambda_0) < \delta, \tag{15}$$

sets $H_f(x_0', \lambda')[t_0, t^*]$ and $\Phi_f(x_0', \lambda')[t_0, t^*]$ are nonempty, compact (in respective spaces), and the multivalued maps

$$(x_0', \lambda') \to H_f(x_0', \lambda')[t_0, t^*] \tag{16}$$

and

$$(x_0', \lambda') \to \Phi_f(x_0', \lambda')[t_0, t^*] \tag{17}$$

upper semicontinuous at the point (x_0, λ_0).

Note, upper semicontinuity for the compact multi-valued map (16) means the following: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all the values of (x_0', λ'), which satisfy condition (17), set $H_f(x_0', \lambda')[t_0, t^*]$ belongs to the ε – neighborhood (in the space $C[t_0, t^*]$) of the set $H_f(x_0, \lambda_0)[t_0, t^*]$. The similar statement is valid also with respect to the multi-valued map (17).

2.2. The principle of invariance and attraction for autonomous systems.

Consider the following autonomous differential equation

$$\dot{x} = f(x) \tag{18}$$

with the function $f : \Omega \to R^n$, defined in some domain $\Omega \subset R^n$. As usually, we put $t_0 = 0$. Within the frames of the given section, for the equation (18) we assume that the following properties take place:

1°. *For any initial state* $x_0 \in \Omega$ *there exusts a local solution;*

2°. Function $f(x)$ *is locally bounded;*

3°. The limit of $x(t)$ *for any sequence of solutions of equation (20) uniformy converging on* $[0, t_1)$ *is the solution of (18) under the condition that* $x(t) \in \Omega$ *for all* $t \in [0, t_1)$.

These properties follow from the main conditions of Section 2.1.

As far as the solution $x(t)$ of equation (18), which is defined on the right maximal interval of existence $[0, \omega)$ is concerned, by $\Lambda^+(x)$ we denote the ω – limit set. Any solution of (18) may be continued onto the right maximal interval of existence $[0, \omega)$. Furthermore, if $\Lambda^+(x) \cap \Omega \neq \emptyset$ then $\omega = +\infty$. The set of equilibrium positions of equation (18) is closed with respect to set Ω (i.e. if Ω is considered as a subspace with the metric induced from R^n). The right uniqueness of solutions is not assumed.

D e f i n i t i o n . *The set* $F \subset \Omega$ *is said to be semi-invariant if for each* $x_0 \in F$ *there exists at least one noncontinuable solution* $x(t)$ *of equation (*18*) with the initial condition* $x(0) = x_0$, *which satisfies the condition* $x(t) \in F$ *for all* $t \in [0, \omega)$.

T h e o r e m 2 .2.1 *Any* ω *– limit set of equation (*18*) is semi-invariant.*

Below for equation (18) we formulate the La-Salle principle of invariance and preset theorems on attraction with the use of a set of Lyapunov functions. One (the main) Lyapunov function provdes for the attraction to the set, on which its derivative turns zero. Auxiliary Lyapunov functions provide for attraction to some set, which may be represented also by the set of nonisolated equilibrium positions. By $w(x)$ we denote an arbitrary function with nonnegative values, which is defined in domain Ω. Put $E(w = 0) @ \{x \in \Omega : w(x) = 0\}$.

T h e o r e m 2 .2.2. *Let for equation (*18*) and for some set* $M \subset \Omega$ *there exists a finite set of locally Lipschitz functions* $V_i(x)$, $i = 0, 1, \ldots, N$, *such that* $D^{*+}V_0(x) \le -w(x)$ *for all* $x \in \Omega$ *and for any* $x \in E(w=0) \setminus \overline{M}$ *there exists* $i \in (1, \ldots, N)$ *such that* $V_i(x) = 0$, $D^{*+}V_i(x) \ne 0$.

Hence for any noncontinuable solution $x(t)$ *of equation (18) the condition* $\Lambda^+(x) \cap \Omega \subset \overline{M}$ *holds.*

Let us speak that the solution $x(t)$ of equation (18) tends to set $F \subset \overline{\Omega}$ if $d(x(t), F) \to 0$ for $t \to \omega$, $t < \omega$, where d is the distance from the point to the set. The solution $x(t)$ weakly tends to set F if there exists a set of points $t_k \to \omega$, $t_k < \omega$, such that $d(x(t_k), F) \to 0$. By $\partial\Omega$ we denote the boundary of set Ω.

C o r o l l a r y 2 .2.1. *Let conditions of Theorem 2.2.2 be satisfied. Hence* $\Lambda^+(x) \subset \overline{M} \cup \partial\Omega$ *and the following statements hold for solutions* $x(t)$ *of equation (*18*):*

1) either $\|x(t)\| \to +\infty$ *or* $x(t)$ *weakly tends to set* $M \cup \partial\Omega$ *for* $t \to \omega$;

2) either $x(t)$ *is unbounded or* $x(t)$ *tends to set* $M \cup \partial\Omega$ *for* $t \to \omega$;

3) if $M \cup \partial\Omega = \emptyset$ *then* $\|x(t)\| \to +\infty$ *for* $t \to \omega$.

C o r o l l a r y 2 .2.2. *Let conditions of Theorem 2.2.2 be satisfied. Hence:*

1) if $\overline{M} \cap \partial\Omega = \emptyset$ *then any bounded solution* $x(t)$ *of equation (18) tends either to set* M *or to set* $\partial\Omega$. *In particular, if* $\Omega = R^n$ *and* M *is a set of equilibrium positions of system (18), then it is dichotomous;*

2) if set $M \subset \Omega$ *is compact than either* $\|x(t)\| \to +\infty$ *or* $x(t)$ *is bounded and tends to* M, *or else* $x(t)$ *leaves any compct set from domain* Ω *for* $t \to \omega$.

For each $x \in \overline{\Omega}$ we introduce the denotation

$$\underline{w}(x) = \begin{cases} w(x), & x \in \Omega, \\ \underline{\lim}_{x' \to x,\, x' \in \Omega} w(x'), & x \in \partial\Omega. \end{cases}$$

Let us speak that function $V(x)$ is continuous up to the boundary when for every point $x \in \partial\Omega$ there exists a finite limit $\lim_{x' \to x,\, x' \in \Omega} V(x')$. Function f is called locally bounded aat the boundary when for any point $x \in \partial\Omega$ function f is bounded at the intersection of some neighborhood x with set Ω.

T h e o r e m 2 .2.3. *Let* $V_0(x)$ *be a locally Lipschitz function continuous up* *to the boundary such that*

$$D^{*+}V_0(x) \le -w(x) \tag{19}$$

for all $x \in \Omega$ *and* f *is the function locally bounded on the boundary.*

Hence $\Lambda^+(x) \subset E(\underline{w} = 0)$ *for any solution of equation (18) defined for all* $t \ge 0$.

T h e o r e m 2 .2.4. *Let* $M \subset \Omega$ *be some set,* $V_0(x)$ *be a locally Lipschitz function contimuous up to the boundary, for which inequality (*19*) holds. Suppose that function* f *is locally bounded on the boundary, and that in some neighborhood of sets* $\overline{\Omega}$ *there are defined continuously differentiable functions* $V_i(x)$, $i = 1, \ldots, N$, *with the property: for any* $x \in E(\underline{w} = 0) \setminus \overline{M}$ *there exists function* V_i *such that* $V_i(x) = 0$ *and the following conditions are satisfied*

$$D^+V_i(x) \ne 0, \text{ if } \; x \in \left(E(w=0) \cap \Omega\right) \setminus \overline{M};$$

$$\left\langle \nabla V_i(x) \cdot \overline{f}(x) \right\rangle > 0 \text{ for all } \overline{f}(x), \text{ if } \; x \in \left(E(\underline{w}=0) \cap \Omega\right) \setminus \overline{M},$$

where $\overline{f}(x)$ *are limit values of function* f *at point* x.

Hence for any solution $x(t)$ *of equation (18) defined for all* $t \ge 0$ *the inclusion* $\Lambda^+(x) \subset \overline{M}$ *holds.*

C o r o l l a r y 2 .2.3. *Let conditions of Theorem 2.3.2 be satisfied. Hence the following statements hold for the solutions of equation (*18*) defined for all* $t \ge 0$:

1) either $\left\| x(t) \right\| \to +\infty$ *or* $x(t)$ *weakly tends to set* M *for* $t \to +\infty$;

2) either the solution $x(t)$ *is unbounded or* $x(t)$ *tends to set* M *for* $t \to +\infty$;

3) if $M = \emptyset$ *then* $\left\| x(t) \right\| \to +\infty$ *for* $t \to +\infty$;

4) if set M *is bounded then either* $\left\| x(t) \right\| \to +\infty$ *or* $x(t)$ *is bounded and tencs to* M *for* $t \to +\infty$.

2.3. Stability.

Application of Lyapunov's method to investigation of stability of equations of motion for mechanical systems in the inexplicit form (6) is bound up with investigation of signdefiniteness of the derivative of the Lyapunov function, which is given inexplicitly, because it contains generalized accelerations. The theorems on stability proved above allow us to weaken the difficulties and suggest a more complete pattern of tracing trajectories in the vicinity of the set of nonisolated equilibrium positions for the systems with friction.

Let us assume that for every point $x \in \Omega$ the cone $\Gamma(x)$, which is represented in the main conditions 1–4 of Section 2.1, possesses the property: $x+\Gamma(x)=\Gamma(x)$. This property holds when $\Gamma(x)$ has some special structure, to be specific: $\Gamma(x)$ represents some nonempty set or coinciding with the whole space R^n, or formed by intersection of semispaces and subspaces of the form

$$L_s^- @ \left\{x'\in R^n : x_s' \le 0\right\},\quad L_s^+ @ \left\{x'\in R^n : x_s' \ge 0\right\},\quad L_s^0 @ \left\{x'\in R^n : x_s' = 0\right\},$$

where the indices s are taken from the set $N(x) @ \left\{s\in(1,\,\ldots,\,n) : x_s = 0\right\}$. Right these sets arise in investigation of systems with friction (8).

Let us put $S_\delta(x) @ \left\{x'\in\Omega : \|x-x'\| < \delta\right\}$, $\Omega_\delta(x) @ S_\delta(x)\cap\Gamma(x)$ and exactly formulate the conditions under which we are going to investigate equation (18).

Let set $\Gamma(x)$ (with the property indicated above) be defined for every point $x\in\Omega$, and let there exist a locally Lipschitz function $V_x:\Omega\to R^+$, which both, as well as function f, satisfy the following conditions:

1. Function f is locally bounded;

2. For any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,\,x)>0$ such that for all $x'\in\Omega_\delta(x)$ the following condition is satisfied $\|f(x)-f(x')\|<\varepsilon$;

3. $f(x)\in\Gamma(x)$ for all $x\in\Omega$;

4. For every fixed point $x\in\Omega$ the following relation holds

$$\left(\forall x'\in\Omega,\ V_x(x')\ge 0\right)\wedge\left(V_x(x')=0 \Leftrightarrow x'\in\Gamma(x)\right) \tag{20}$$

and if $\Gamma(x)\ne R^n$ then there exist such numbers $\alpha=\alpha(x)>0$ and $\delta=\delta(x)>0$, that

$$D^{*+}V_x(x')<-\alpha \tag{21}$$

for all $x'\in S_\delta(x)\setminus\Gamma(x)$.

It follows from (20) and (21) that for any $x\in\Omega$ and under the appropriate choice of the number $\delta>0$ set $\Omega_\delta(x)$ possesses the property of absolute sector, i.e. the trajectory of any solution with the initial condition $x(0)\in\Omega_\delta(x)$ stays in $\Omega_\delta(x)$ for all $t\ge 0$ for which $x(t)\in S_\delta(x)$.

D e f i n i t i o n . *Set* $\Omega_\delta(x)$ *is called the* Γ *– sector with the node* x *and radius* δ (*furthermore, condition (23) holds for the number* $\delta>0$). In the case, when $\Gamma(x)=R^n$, any arbitrary positive number is considered as the radius.

Consider a compact set $M\subset\Omega$, which for each $x\in M$ satisfies the condition

$$M \subset \Gamma(x). \tag{22}$$

Introduce the denotation $M^{\beta} @ \{x' \in R^n : d(x', A) < \beta\}$ for any arbitrary number $\beta > 0$.

If V is a real-valued function defined within some neighborhood of set M, and γ is a number, then $E(V < \gamma) @ \{x : V(x) < \gamma\}$. The set $E(V = \gamma)$ is defined similarly.

T h e o r e m 2 .3.1. *Let a nonnegative locally Lipschitz function $V(x)$ with the following properties be defined in some neighborhood M^{ρ}, $\rho > 0$, of set M:*

1) $V(x) = 0 \Leftrightarrow x \in M$;

2) for any Γ – sectors $\Omega_{\delta}(x)$ with the node $x \in M$ and radius $\delta < \rho$ the condition $D^{*+}V(x') \le 0$ is satisfied for all $x' \in \Omega_{\delta}(x)$.

Hence for any $\varepsilon > 0$ and $\tau > 0$ there exist $\delta > 0$ and a finite coverage of set M with Γ – sectors $\Omega_{\delta_i}(x_i)$, $x_i \in M$, $i = 1, \ldots, m$, such that any solution $x(t)$ with the initial condition $x(0) \in M^{\delta}$ is defined for all $t \ge 0$ and satisfies the condition $x(t) \in M^{\varepsilon}$ for all $t \ge 0$ and

$$\forall t \ge \tau, \;\; x(t) \in \left\{ \bigcup \Omega_{\delta_i}(x_i) : i \in (1, \ldots, n) \right\} \tag{23}$$

Tracing trajectories for equation (20) within the limits of the Γ – sector $\Omega_{\delta}(x)$ may be substantially simplified, what allows to investigate signdefiniteness of the derivative of function V more efficiently.

2.4. Asymptotic stability and instability.

Note, Theorem 2.3.1 states not only the fact of stability of set M but also satisfaction of condition (23), from which it follows that function $V(x(t))$ is nonincreasing for all $t \ge \tau$ along any solution $x(t)$ with the initial condition $x(0) \in M^{\delta}$. The latter side by side with the principle of invariance may be used in the investigation of asymptotic stability of set M.

T h e o r e m 2.4.1. *Let conditions of Theorem 2.3.1 be satisfied, and, additionally, $D^{*+}V(x') < 0$ for all $x' \in \Omega_{\delta}(x) \setminus M$. Hence M is asymptotically stable (i.e. M is stable and $d(x(t), M) \to 0$ for $t \to +\infty$ for any solution $x(t)$ with the initial value $x(0) \in M^{\delta}$).*

T h e o r e m 2 . 4 . 2 . *Let conditions of Theorem 2.4.1 be satisfied, and, additionally, set* $E\left(D^{*+}V=0\right)\bigcap M^{\rho}$ *do not contain any closed semi-invariant sets, which do not intersect with* M *and belong to some covering of set* M *with* Γ *– sectors. Hence set* M *is asymptotically stable.*

If, otherwise, $\Gamma(0)\neq\{0\}$ *and condition 2 of Theorem 3.2.1. is replaced with the condition:*

2') for any Γ *– sector* $\Omega_{\delta}(x)$ *with the node* $x\in M$ *and radius* $\delta<\rho$ *the inequality* $D^{*+}V(x')\geq 0$ *holds for all* $x'\in\Omega_{\delta}(x)$*, then, under the same additional assumption, set* M *is unstable.*

3 EQUATIONS OF DYNAMICS FOR MECHANICAL SYSTEMS WITH SLIDING FRICTION

3.1. Solvability of equations of motion with respect to generalized accelerations.

The conditions of solvability of equations of motion with respect to velocities are determined by the methods of investigation. In the present work, the principle of contracting mappings has been laid as the basis of the approach to solvability of equations of motion with respect to accelerations. The respective method presumes existence of definite properties, which are not represented in equations (6): from the definition of frictions forces $Q^{T}(t,\,q,\,\dot q,\,\ddot q)$ by formula (10) one cannot derive even the fact of their continuity with respect to variable $\ddot q$.

For the purpose of overcoming the difficulties that arise in this case, we introduce in consideration a new system of equations, which is close in its structure to the initial equations of motion and which is henceforth called *equations of dynamics.* These equations, generally speaking, differ from the initial equations (8), but, nevertheless, under definite conditions these define the same equations in explicit form and, so, have the same solutions. Introduce the denotation:

$$\mathrm{N}\,(\dot q) @ \left\{s\in(1,\,\dots,\,k_*):\dot q^{s}=0\right\}$$

$$\mathrm{N}_0(t,\,q,\,\dot q,\,\ddot q) @ \left\{s\in N(\dot q)\,:\left|Q_s^{T0}(t,q,\dot q,\ddot q)\right|\leq f_s\left(t,\,q^{s},0\right)\left|N_s(t,\,q,\,\dot q,\,\ddot q)\right|\right\}$$

and write down the following system of equations:

$$\begin{cases} \sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = Q_s^{T0}(t,q,\dot{q},\ddot{q}) + g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}), \\ \qquad s \in \mathrm{N}\,(t,q,\dot{q},\ddot{q}), \\ \sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = f_s(t,q^s,0)\left| N_s(t,q,\dot{q},\ddot{q}) \right| sgn\, Q_s^{T0}(t,q,\dot{q},\ddot{q}) + \\ \qquad + g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}),\; s \in \mathrm{N}\,(\dot{q}) \setminus \mathrm{N}_0(t,q,\dot{q},\ddot{q}), \\ \sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = f_s(t,q^s,\dot{q}^s)\left| N_s(t,q,\dot{q},\ddot{q}) \right| sgn\, \dot{q}^s + \\ \qquad + g_s(t,q,\dot{q}) + Q_s^A(t,q,\dot{q}),\; s \in (1,\ldots,k_*) \setminus \mathrm{N}\,(\dot{q}), \\ \sum_{i=1}^{k} a_{si}(t,q)\ddot{q}^i = g_s(t,q^s,\dot{q}) + Q_s^A(t,q,\dot{q}),\; s = k_* + 1,\ldots,k. \end{cases} \tag{24}$$

L e m m a 3 .1.1. *If for all* $s \in \mathrm{N}\,(\dot{q}) \setminus \mathrm{N}_0(t, q, \dot{q}, \ddot{q})$, $(t,q,\dot{q},\ddot{q}) \in \Omega \times R^k$, *such that* $\left| N_s(t,q,\dot{q},\ddot{q}) \right| \neq 0$ *the inequality*

$$f_s(t,q,0)\left| \frac{\partial \left| N_s(t, q, \dot{q}, \ddot{q}) \right|}{\partial \ddot{q}^s} \right| < a_{ss}(t, q), \tag{25}$$

holds, then the equation of motion (6) with friction forces (10) is equivalent to the system of equations (24) (i.e. their solutions coincide in some or another sense).

Unlike that in equation (6), right-hand sides of equations (24) are continuous with respect to $\ddot{q}$ for any fixed $(t, q, \dot{q}) \in \Omega$.

Let $\left[A(t, q) \right](k_*)$ be a submatrix of matrix $A(t, q)$, which has been obtained from it by deleting the first k_* rows and columns. Like $A(t, q)$ $\left[A(t, q) \right](k_*)$ is a positive definite and, consequently, nonsingular matrix. By $\left[\tilde{a}_{k_*+i,\, k_*+j} \right]_1^{k-k_*}$ we denote the matrix inverse with respect to $\left[A(t, q) \right](k_*)$. Consider the inequalities

$$\left| a_{s\nu}\gamma_{s\nu} - f_s \frac{\partial \left| N_s \right|}{\partial \ddot{x}^\nu} e_s + \sum_{j=k_*+1}^{k} \left(a_{sj} - f_s \frac{\partial \left| N_s \right|}{\partial \ddot{x}^j} e_s \right) \sum_{i=k_*+1}^{k} \tilde{a}_{ij}\, a_{i\nu} \right| < \frac{a_{ss}}{k_*} \tag{26}$$

for all $s, \nu = 1, \ldots, k_*$ at each point $(t, q, \dot{q}, \ddot{q}^*) \in \Omega \times R^{k_*}$ such that $\left| N_s \right| \neq 0$, where e_s may assume the values $+1$ or -1 and $\gamma_{s\nu} = \begin{cases} 0, & s = \nu; \\ 1, & s \neq \nu. \end{cases}$

Since the inequality $a_{ss}(t, q) > 0$ holds in the domain under scrutiny, inequalities (26) are always satisfied for sufficiently small functions f_s and for offdiagonal elements a_{sv} of matrix A. Hence, the following statement is valid.

T h e o r e m 3 .1.1. *Let inequalities (*26*) hold. Hence equations of dynamics (*24*) aare unequivocally solvable with respect to* $\ddot{q}$ *and may be reduced to the form*

$$\ddot{q} = G(t, q, \dot{q}). \tag{27}$$

From now on inequalities (26) are assumed to be satisfied (below we use a strengthened variant of these inequalities), and function G, which is the slution of equations (24) with respect to $\ddot{q}$, is assumed to be defined on set Ω.

3.2. The properties of function *G*.

Introduce the following denotations for every point $(t, q, \dot{q}, \ddot{q}) \in \Omega \times R^k$:

$$\Gamma = \Gamma(t, q, \dot{q}, \ddot{q}) @ \Big\{ \dot{q}' \in R^k : \dot{q}^{s\prime} = 0, \text{ если } s \in \mathrm{N}(\dot{q}), \ f_s \,|\, N_s \,| > \big| Q_s^{T0} \big|;$$
$$\dot{q}^{s\prime} Q^{T0} \le 0, \text{ если } s \in \mathrm{N}(\dot{q}), \ f_s |\, N_s \,| \le \big| Q_s^{T0} \big|, \ |\, N_s \,| \ne 0 \Big\}.$$

If $\mathrm{N}(\dot{q}) = \emptyset$ or $|\, N_s \,| = 0$ for all $s \in \mathrm{N}(\dot{q})$, then we assume $\Gamma @ R^k$.

Suppose that

$$S_\delta = S_\delta(t, q, \dot{q}) @ \big\{ (t', q', \dot{q}') \in \Omega : t \le t' < t + \delta, \ \| q - q' \| < \delta, \ \| \dot{q} - \dot{q}' \| < \delta \big\},$$
$$\Omega^0 = \Omega^0(t, q, \dot{q}) @ \big\{ (t', q', \dot{q}') \in \Omega : \dot{q}' \in \Gamma(t, q, \dot{q}, G) \big\},$$
$$\Omega^0_\delta(t, q, \dot{q}) @ \Omega^0(t, q, \dot{q}) \cap S_\delta(t, q, \dot{q}).$$

Let us speak that *the strengthened inequalities* (26) hold when their right-hand sides are replaced with the expression $\gamma / k_* n$, where $\gamma = \gamma(t, q, \dot{q})$ is a continuous function, which satisfies the condition $0 < \gamma(t, q, \dot{q}) < 1$ for all $(t, q, \dot{q}) \in \Omega$. Let us put $n = 1$ when the set $\mathrm{N}(\dot{q}) = \emptyset$ or it contains not more than 2 indices, and $n = k - 1$ when $\mathrm{N}(\dot{q})$ contains $k > 2$ indices.

T h e o r e m 3 .2.1. *Let the strengthened inequalities (28) be satisfied. Hence for every point* $(t, q, \dot{q}) \in \Omega$ *defined are set* $\Gamma(t, q, \dot{q}, G)$, *which represents an intersection of subspaces and semispaces of generalized velocities* $\dot{q}$, *the Lipschitz function* $V_{(t, q, \dot{q})}(\dot{q}')$ *and (at each point* $(t, q, \dot{q}) \in \Omega$*) the following properties of function* G *hold:*

1. G *is locally bounded;*
2. G *is continuous at the point* $(t, q, \dot{q}) \in \Omega$ *along the set* $\Gamma(t, q, \dot{q}, G)$*;*

3. $G(t,\, q,\, \dot{q}) \in \Gamma(t,\, q,\, \dot{q},\, G)$*;*

4. Function $V_{(t,q,\dot{q})}(\dot{q}')$ *is nonnegative and satisfies the condition*

$$V_{(t,\, q,\, \dot{q})}(\dot{q}') = 0 \Leftrightarrow \dot{q}' \in \Gamma(t,\, q,\, \dot{q},\, G)$$

and if $\Gamma(t,\, q,\, \dot{q},\, G) \neq R^k$*, then there exist numbers* $\alpha = \alpha(t,\, q,\, \dot{q}) > 0$*,* $\delta = \delta(t,\, q,\, \dot{q}) > 0$*, such that* $D^+ V_{(t,\, q,\, \dot{q})}(\dot{q}') < -\alpha$ *for all* $(t', q', \dot{q}') \in S_\delta(t,\, q,\, \dot{q}) \setminus \Omega^0(t,\, q,\, \dot{q})$.

4 RIGHT-HABD SIDE SOLUTIONS OF EQUATIONS OF THE SYSTEM DYNAMICS WITH FRICTION

4.1. Existence and general properties of solutions.

Let us introduce the denotations: $x = (q,\, \dot{q})$, $f = (G_1,\, G_2)$, where $G_1(t,\, x) = \dot{q}$, $G_2(t,\, x) = G(t,\, q,\, \dot{q})$. Hence equation (29) writes in the form of equation (13) investigated in Part 2 of the present paper, and from Theorem 3.2.1 it follows that the principal conditions of Section 2.1 are satisfied. Indeed, cone $\Gamma(t,\, q,\, \dot{q},\, G)$ (under the condition that $\ddot{q} = G(t,\, q,\, \dot{q})$) is defined as an intersection of subspaces and semispaces of generalized velocities $\dot{q}^s$ (at points of discontinuity $\dot{q}^s = 0$ of function G). When considering the Cartesian product $R^k \times \Gamma(t,\, q,\, \dot{q},\, G)$, we obtain a closed cone $\Gamma(t,\, x)$ (the denotation remains the same) in space R^{2k}, for which the following obvious equality $x + \Gamma(t,\, x) = \Gamma(t,\, x)$ holds at each point $(t,\, x) \in \Omega$, where $\Omega \subset R^{2k+1}$ is the domain of definition for the right-hand sides of dynamics equations (26).

Consider formulations of the principal theorems.

T h e o r e m 4 .1.1. *For any initial state* $(t_0,\, q_0,\, \dot{q}_0) \in \Omega$ *there exists a local right solution of problem (29). Any right solution is* R*-right on its interval of definition, i.e. the right derivative* $D^+\dot{q}(t)$ *is a function continuous on the right.*

T h e o r e m 4 .1.2. *The limit предел* $(q(t),\, \dot{q}(t))$ *of the sequence of right solutions of dynamics equations (29), which is homogeneous on the segment* $[t_0,\, t_0 + a)$ *and satisfies the condition* $(t,\, q(t),\, \dot{q}(t)) \in \Omega$ *for* $t \in [t_0,\, t_0 + a)$*, represents the right solution of the dynamics equations.*

Let us speak that the right solution $(q(t),\, \dot{q}(t))$, which is defined on the right maximal segment of existence $[t_0,\, \omega)$, tends to the boundary of set Ω if for any compact set $K \subset \Omega$ there exists a point $t_K \in (t_0, \omega)$ such that $(t, q(t), \dot{q}(t)) \notin K$ for all $t \in (t_K, \omega)$ (the solution leaves compact subsets from Ω).

T h e o r e m 4 .1.3. *Any right solution of dynamics equations (29) with the initial conditions* $(t_0, q_0, \dot{q}_0) \in \Omega$ *may be continued onto the right maximal segment of existence* $[t_0, \omega)$. *Any right noncontinuable right-hand-side solution tends to the boundary of set* Ω. *Furthermore:*

1) if $\Omega = (a, b) \times H$ *and* $\omega < b$, *then* $(q(t), \dot{q}(t))$ *tends to the boundary of set* H*;*

2) if $\Omega = R^1 \times H$, *then either* $\omega = +\infty$ *or* $\omega < +\infty$ *and* $(q(t), \dot{q}(t))$ *tends to the boundary of* H*;*

3) if $\Omega = R^1 \times R^{2k}$, *then either* $\omega = +\infty$ *or* $\omega < +\infty$ *and*

$$\| q(t) \| + \| \dot{q}(t) \| \to +\infty \text{ for } t \to \omega - 0.$$

4.2. Continuous dependence of solutions on initial states and parameters.

Let functions f_s, $|N_s|$, g_s^A, Q_s^A, a_{si} be dependent also on parameter λ, which assumes its values in some metric space Λ and are continuous with respect to a set of arguments (denotations for them will be still the same). Matrix $A(t, q, \lambda)$ is assumes to be positive definite and symmetric in the domain of definition of its variables.

Consider the initial-value problems

$$\ddot{q} = G(t, q, \dot{q}, \lambda), \; q(t_0) = q_0, \; \dot{q}(t_0) = \dot{q}_0, \; \lambda = \lambda_0, \tag{28}$$

where function $G(t, q, \dot{q}, \lambda)$ is defined from the equations of dynamics.

T h e o r e m 4 .2.1. *Let equation (28) possess the property of right uniqueness,* $(q(t), \dot{q}(t))$ *is the right solution of problem (28) with the data* $(t_0, q_0, \dot{q}_0, \lambda_0)$ *and with the right maximal segment of existence* $[t_0, \omega)$. *Hence for any* $\varepsilon > 0$ *and* $t^* \in (t_0, \omega)$ *there exists* $\delta > 0$ *such that any right solution* $(q'(t), \dot{q}'(t))$ *of problem (28) with the initial states* $(t_0', q_0', \dot{q}_0')$ *and with the values of* λ', *which satisfy the conditions*

$$\| q_0' - q_0 \| < \delta, \quad \| q' - \dot{q}_0 \| < \delta, \quad t_0 - \delta < t_0' \le t_0, \quad d(\lambda', \lambda_0) < \delta,$$

may be continued onto the right maximal interval of existence $[t_0', \omega')$, $\omega' > t^*$ *and* *for the continuation* $\left(q'(t),\, \dot{q}'(t)\right)$ *the condition*

$$\| q'(t) - q(t) \| < \varepsilon, \quad \| \dot{q}'(t) - \dot{q}(t) \| < \varepsilon$$

holds for all $t \in \left[t_0,\, t^*\right]$.

C o r o l l a r y 4 .2.1. *Let equation (28) possess the property of right uniqueness,* $\lambda_n \to \lambda_0$, $\left(t_{n0},\, q_{n0},\, \dot{q}_{n0}\right) \to \left(t_0,\, q_0,\, \dot{q}_0\right)$ $t_{n0} \le t_0$, *and* $\left(q'(t),\, \dot{q}'(t)\right)$ *is the right solution of problem (28) with the data* $\left(t_0,\, q_0,\, \dot{q}_0,\, \lambda_0\right)$, *which is defined on the right maximal interval of existence* $\left[t_0,\, \omega\right)$. *Hence for any* $t^* \in \left(t_0,\, \omega\right)$ *on the segment* $\left[t_0,\, t^*\right]$, *the sequence of right solutions* $\left(q^n(t),\, \dot{q}^n(t)\right)$ *of equation (28) с данными* $\left(t_0,\, q_{n0},\, \dot{q}_{n0},\, \lambda_n\right)$ *is defined and uniformly converhes to* $\left(q(t),\, \dot{q}(t)\right)$, *while beginning from some* n.

Without any assumption of right uniqueness for equations of dynamics it is possible to formulate some theorems on the upper semi-continuous dependence of solutions on the initial system‘s states and parameters.

R e m a r k . One of the principal directions in the theory of differential inclusions with discontinuous right-hand sides (including also the equations of dynamics of scrutinized systems with friction) is bound up with their representation in the form of differential inclusions. In case if the transition to the differential inclusion would not lower the exactness of the initial problem’s statement, then it would be possible to automatically disseminate onto systems with friction any known facts from the theory of differential inclusions, which is well developed presently. For example, it would be possible to extend Kneser’s theorem (which is related to connectedness of a set of solutions and is known for differential inclusions) onto equations of dynamics of systems with friction. This may be done in the case when normal reactions are independent of generalized accelerations. Sets of solutions of Karateodore equations for dynamics of systems with friction and of the differential inclusions formed by the simplest convex extension of the definition in the sense of A.F. Filippov [33] for the right-hand sides of equations of dynamics at points of discontinuity coincide. Any solutionof the differential inclusion is right-hand side.

4.3. Example of P. Painlevé.

Let us turn back to the example of P. Painlevé and write down conditions of solvability (28) for it from section 2.1. Equations of motion write:

$$\begin{cases} 2\ddot{x} - r\sin\theta\, \ddot{\theta} = r\dot{\theta}^2 \cos\theta + Q_1^T \\ -r\sin\theta\, \ddot{x} + r^2\ddot{\theta} = rg\cos\theta \end{cases} \tag{29}$$

The absolute value of the normal reaction and the friction at rest:

$$\left| N_1\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right| = \left| r\left(\ddot\theta\cos\theta - \dot\theta^2\sin\theta\right) - 2g\right|,$$
$$Q_1^{T0}\left(\theta,\, \dot\theta,\, \ddot\theta\right) = -r\sin\theta\,\ddot\theta - r\dot\theta^2\cos\theta\,.$$

In the general case, the fricxtion forces write:

$$Q_1^T\left(\theta,\, \dot x,\, \dot\theta,\, \ddot\theta\right) = \begin{cases} Q_1^{T0}(\theta,\dot\theta,\ddot\theta),\ \text{if}\ \ \dot x = 0\ \ \text{and} \\ \qquad\qquad \left|Q_1^{T0}\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right| \le f\left|N_1\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right|; \\ f\left|N_1\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right| sgn\, Q_1^{T0}\left(\theta,\, \dot\theta,\, \ddot\theta\right),\ \text{if}\ \ \dot x = 0\ \ \text{and} \\ \qquad\qquad \left|Q_1^{T0}\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right| > f\left|N_1\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right|; \\ -f\left|N_1\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right| sgn\,\dot x,\ \text{if}\ \ \dot x \ne 0. \end{cases} \tag{30}$$

where $f > 0$ is the friction coefficient (constant). Since $\left|N_1\left(\theta,\, \dot\theta,\, \ddot\theta\right)\right|$ is independent of $\ddot x$, the dynamics equations (of the form (24)) in the given case coincide with equations of motion (29), while presuming extension of the definition of friction forces with use of formula (30).

Consider inequalities (26) in connection with equations (29). Here $k = 2$, $k_* = 1$, $\nu = 1$, $\tilde a_{22} = 1/r^2$. The sets

$$\mathrm{N} = \begin{cases} \{1\}, & \text{if}\ \ \dot x = 0 \\ \emptyset, & \text{if}\ \ \dot x \ne 0 \end{cases}$$

$$\mathrm{N}_0 = \begin{cases} \{1\},\ \text{if}\ \ \dot x = 0, & \left|Q_1^{T0}\right| \le f\left|N_1\right| \\ \emptyset,\ \ \text{if}\ \left(\dot x \ne 0\right)\vee\left(\dot x = 0,\ \left|Q_1^{T0}\right| > f\left|N_1\right|\right) \end{cases}$$

Since $\partial_{\ddot\theta}\left|N_1\right| = r\cos\theta\, sgn N_1$, if $N_1 \ne 0$ then formula (26) has the form of inequality $sin^2\theta + f\left|\cos\theta\,\sin\theta\right| < 2$ or, equivalently,

$$f\left|\sin\theta\cos\theta\right| < 1 + \cos^2\theta\,. \tag{31}$$

When inequality (31) holds, equations (29) are unequivocally solvable with respect to accelerations, and so saved from any contradictions with Coulomb's laws of friction, which was emphasized by P. Painlevé. Noteworthy, as far as the angle $\theta_0 \in \left(0,\, \pi/2\right)$ and the friction coefficient f, which satisfy the inequality

$$f\sin\theta_0\cos\theta_0 > 1 + \cos^2\theta\,, \tag{32}$$

for definite initial states, either nomotion undergoes the laws of friction accepted or already two motions undergo these laws. Inequalities (31) and (32) are consistent with the results of analysis of the P. Painlevé's example (see [17]). It may readily be verified that inequality (31) holds for any $\theta \in [0,\ 2\pi]$, when the friction coefficient f does not exceed the value, which approximately is 2.8.

5 ATTRACTION AND STABILITY OF THE SET EQUILIBRIUM POSITIONS FOR THE SYSTEMS WITH FRICTION

5.1. Attraction for autonomous systems.

The section is devoted to investigation of issues of attraction of the set of equilibrium positions of system (6) under the effect of potential, dissipative, gyroscopic forces and forces of sliding friction in the autonomous case. A general scheme of investigations releted to attraction for the mechanical system under scrutiny is proposed. Its efficiency is demonstrated by an example.

Now consider equations of motion of a mechanical system (6) under the condition that the kinetic energy and the forces acting on the system are independent of

time. As before, in this case we assume that the system's kinetic energy may be represented as the sum $T_a = T + T_1 + T_0$ of the positive definite quadratic form T of generalized velocities with a symmetric positive definite matrix $A(q) = \left[a_{ij}(q)\right]_1^k$, a linear form of generalized velocities $T_1 = \sum_{i=1}^{k} a_i(q)\dot{q}^i$ and a function $T_0(q)$. Suppose further that

$$Q_s^A(q,\ \dot{q}) = D_s(q,\ \dot{q}) + K_s(q),$$

and introduce the denotation

$$\Gamma_s(q,\dot{q}) @ \frac{\partial T_1}{\partial q^s} - \frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}^s} = \sum_{j=1}^{k}\left(\frac{\partial a_j}{\partial q^s} - \frac{\partial a_s}{\partial q^j}\right)\dot{q}^j,\quad Q_s^e(q) \overset{\Delta}{=} \frac{\partial T_0}{\partial q^s},$$

where $K_s(q) = -\partial\Pi/\partial q^s$, $\Pi(q)$ is the system's potential energy, $D_s(q,\ \dot{q})$ are the dissipative forces, $\Gamma_s(q,\ \dot{q})$ are gyroscopic forces with conditions $D_s(q,\ 0) \equiv 0$, $\Gamma_s(q,\ 0) \equiv 0$; $Q_s^e(q)$ are generalized transferable inertia forces $(s = 1,\ \ldots,\ k)$. Hence the initial system writes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = D_i + \Gamma_i + K_i + Q_i^e + Q_i^T,\quad i = 1,\ \ldots,\ k. \tag{33}$$

Put $V_0 @ \Pi + T - T_0$. Having multiplies equations (33) by $\dot{q}^i$ and then adding them, we obtain

$$D^+V_0(q, \dot{q}) = -\sum_{i=1}^{k_*} f_i |N_i||\dot{q}^i| + \sum_{i=1}^{k} D_i(q, \dot{q})\dot{q}^i \ (\leq 0) \tag{34}$$

Let $J \subset (1, \ldots, k_*)$. Introduce the denotations

$$w_J(q, \dot{q}) = \sum_{i \in J} f_i |N_i||\dot{q}^i|$$

$$H_J = \left\{(q, \dot{q}) : \dot{q}^i = 0, \ i \in J\right\},$$

$$M_J = \left\{(q, \dot{q}) : \dot{q}^i = 0, \ f_i |N_i| \geq |Q_i^{T0}|, \ i \in J\right\},$$

$$M = \left\{(q, 0) : f_i \left| N_i \right| \geq \left| K_i + Q_i^e \right|, i = 1, \ldots, k_*; \right.$$
$$\left. K_i + Q_i^e = 0, \ i = k_* + 1, \ldots, k\right\} \tag{35}$$

Obviously, if $J' \subset J$ then $w_{J'} \leq w_J \leq -D^+V_0$, $H_J \subset H_{J'}$, $M_J \subset M_{J'}$, and it is always valid that $M \subset M_J \subset H_J \subset E\ (w_J = 0)$.

Set M defined by equality (35), represents a set of equilibrium positions for the equations (33). Sets H_j and M are closed.

Likewise before we assume that

$N(\dot{q}) = \left\{i \in (1, \ldots, k_*) : \dot{q}^i = 0\right\}$.

It can easily be verified that

$(q, \dot{q}) \in H_J \Leftrightarrow J \subset N(\dot{q})$

and

$$\left((q, \dot{q}) \in E(w_J = 0) \setminus H_J\right) \Leftrightarrow \left(J \setminus N(\dot{q}) \neq \emptyset\right) \wedge \left(\forall i \in J \setminus N(\dot{q}), \ \left|N_i\right| = 0\right) \tag{36}$$

Directly from above assumptions and from Theorem 2.2.1 it follows that for any solution $z(t) = (q(t), \dot{q}(t))$ of equation (33) and for the set $J \subset (1, \ldots, k_*)$ the following inclusion

$$\Lambda^+(z) \subset E(w_J = 0) \tag{37}$$

holds.

T h e o r e m 5 . 1 . 1 . *Let for some set* $J \subset (1, \ldots, k_*)$ *there exists a finite set of locally Lipschitz functions* $V_i(q, \dot{q})$ $(i = 1, 2, \ldots, N)$ *such that*

$$\left(J \setminus N(\dot{q}) \neq \emptyset\right) \wedge \left(\forall j \in J \setminus N(\dot{q}),\ \ \left|N_j\right| = 0\right) \Rightarrow \\ \Rightarrow \left(\exists i \in \left(\overline{1,\ N}\right),\ V_i = 0,\ D^{*+}V_i \neq 0\right) \tag{38}$$

Hence $\Lambda^+(z) \subset H_J$ *for any solution* $z(t)$ *of equation (33).*

Т е о р е м а 5 .1.2. *Let for the set* $J \subset (1,\ \ldots,\ k_*)$ *condition (*38*) holds. Hence for any solution* $z(t)$ *of equation (*33*) the following condition*

$$\Lambda^+(z) \subset M_J \tag{39}$$

holds, and, furthermore:

1) either $\left\| z(t) \right\| \to \infty$ *or* $M_J \neq \emptyset$ *and* $z(t)$ *weakly tends to* M_j*;*

2) either solution $z(t)$ *is unbounded or* $M_J \neq \emptyset$ *and* $z(t)$ *tends to* M_j.

T h e o r e m 5 .1.3. *Let condition (40) holds for the set* $J \subset (1,\ \ldots,\ k_*)$*, and the dissipation is complete with respect to* $\dot{q}^{k_*+1},\ \ldots,\ \dot{q}^k$*, i.e.*

$$\sum_{i=1}^{k} D_i(q,\ \dot{q})\dot{q}^i \leq -\gamma \sum_{i=k_*+1}^{k} \dot{q}^{i2} \tag{40}$$

for some $\gamma > 0$. *Hence for any solution* $z(t)$ *of equation (33) the following inclusion*

$$\Lambda^+(z) \subset M \tag{41}$$

holds.

Furtermore:

1) either $\left\| z(t) \right\| \to \infty$ *or* $M \neq \emptyset$ *and* $z(t)$ *weakly tends to* M *;*

2) either решение $z(t)$ *is unbounded or* $M \neq \emptyset$ *and* $z(t)$ *tends to* M *;*

3) $M = \emptyset \Leftrightarrow \left\| z(t) \right\| \to \infty$ *for any solution* $z(t)$ *of equation (33).*

Note, within the frames of assumptions of Theorem 5.1.3 system (33) is dichotomous. In order to reveal this fact under condition (40) or under the condition that $k = k_*$ it is sufficient to verify relation (38), while putting $J \subset (1,\ \ldots,\ k_*)$. The latter means the following: for any index $J \subset (1,\ \ldots,\ k_*)$ such that

$$\dot{q}^j \neq 0,\quad \left|N_j\right| = 0 \tag{42}$$

there must exist some function V_i (from the finite set of locally Lipschitz functions) at some point $(q,\ \dot{q})$ such that

$$V_i(q,\ \dot{q}) = 0,\ \ D^{*+}V_i(q,\ \dot{q}) \neq 0 \tag{43}$$

Functions, which satisfy conditions (43), may be defined in the process of analysis of conditions (42). Let us demonstrate this with the use of an example.

5.2. Example. A pendulum system with friction in the bearing and the hinge.

Consider a flat mechanical system, which consists of a piston B of mass m_1 moving with sliding friction along a straight tube Ox, which is inclined at an angle $\alpha = const$ $(0 \le \alpha < \pi/2)$ with respect to the horizontal plane, with the coordinate $x = q^1$; and a heavy perfectly rigid body of mass m_2 revolving with friction around a cylindric hinge, installed on the piston and placed at a distance r from the hinge to the mass center C. J_C is the inertia moment with respect to the mass center (Fig. 1). Angle β of deviation of BC from the normal to Ox, which is directed downwards, is assumed to be q^2. The friction coefficients – f_1 in the piston and f_2 in the hinge – are assumed to be constant, $m = m_1 + m_2$, $J = J_C + m_2 r^2$.

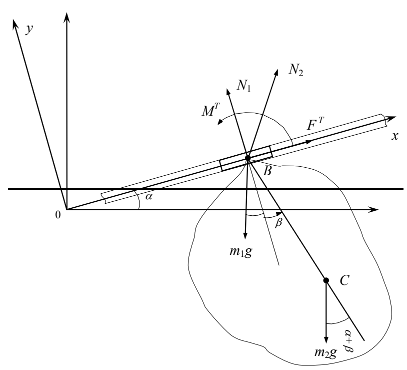

Figure 1. The pendulum system with friction in the bearing and in the hinge

Equations of the system in Lagrange form may be written in the following form:

$$\begin{cases} m\ddot{x} + m_2 r \cos\beta\,\ddot{\beta} = m_2 r\dot{\beta}^2 \sin\beta - mg\sin\alpha + Q_1^T \\ m_2 r\cos\beta\,\ddot{x} + J\ddot{\beta} = -m_2 gr\sin(\alpha+\beta) + Q_2^T. \end{cases} \tag{44}$$

The modules of the normal reactions and the generalized friction forces for the case of relative equilibrium with respect to x and β write as follows:

$$| N_1 | = \left| m_2 r\left(\ddot{\beta}\sin\beta + \dot{\beta}^2\cos\beta\right) + mg \right|,$$

$$| N_2 | = m_2\left[\left(\ddot{x} + r\ddot{\beta}\cos\beta - r\dot{\beta}^2 sin\beta\right)^2 + \left(r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g\right)^2\right]^{1/2},$$

$$Q_1^{T0} = m_2 r\left(\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta\right) + mg\sin\alpha \ \left(\dot{x} = 0,\ \ddot{x} = 0\right),$$

$$Q_2^{T0} = m_2 r\left(\ddot{x}\cos\beta + g\sin\left(\alpha + \beta\right)\right) \ \left(\dot{\beta} = 0,\ \ddot{\beta} = 0\right).$$

The generalized friction forces are defined for $s = 1,\ 2$ by the formula

$$Q_s^T = \begin{cases} Q_s^{T0}, & \text{if } \dot{q}^s = 0,\ \left|Q_s^{T0}\right| \le f_s \,| N_s |_{\ddot{q}^s = 0} \\ f_s \,| N_s |\, sgn\, Q_s^{T0}, & \text{if } \dot{q}^s = 0,\ \left|Q_s^{T0}\right| > f_s \,| N_s |_{\ddot{q}^s = 0} \\ -f_s \,| N_s |\, sgn\, \dot{q}^s, & \text{if } \dot{q}^s \ne 0. \end{cases}$$

The sufficient condition for satisfaction of inequalities (26) and (25) has the form:

$$m_2 r\left(| \cos\beta | + f_1 | \sin\beta |\right) < m/2,$$

$$m_2\left(r| \cos\beta | + f_2\right) < J/2,$$

$$m_2 r\, f_2\left(| \cos\beta | + | \sin\beta |\right) < J/2.$$

The set of equilibrium positions for the system writes:

$$M = \left\{\left(q,\ \dot{q}\right) : \dot{x} = 0,\ \dot{\beta} = 0,\ f_1 \ge \mathrm{tg}\alpha,\ \ f_2 \ge r\left|\sin\left(\alpha + \beta\right)\right|\right\}$$

where $f_1,\ f_2$ are friction coefficients (constant values) in the piston and the hinge, respectively.

In the capacity of the principal Lyapunov function, as it follows from the general Theorem 1.3, we take the system's energy:

$$V_0 = T + \Pi = 1/2\left(m\dot{x}^2 + 2m_2 r\dot{x}\dot{\beta}\cos\beta + J\dot{\beta}^2\right) +$$
$$+ mgx\sin\alpha + m_2 gr\left(1 - \cos\left(\alpha + \beta\right)\right).$$

Consider the following auxiliary Lyapunov functions for the set of indices $J_0 = \{1,\ 2\}$

$$V_1 = \dot{x},\ \ V_2 = \dot{\beta},\ \ V_3 = r\dot{\beta}^2 + g\cos\left(\alpha + \beta\right),\ \ V_4 = \sin\left(\alpha + \beta\right)$$

and the function

$$w = f_1 |N_1||\dot{x}| + f_2 |N_2||\dot{\beta}| = -D^+V_0$$

and let us show that condition (40) is satisfied for them.

There are the following 3 possibilities of satisfaction of the condition

$$\left(J_0 \setminus \mathrm{N}(\dot{q}) \neq \emptyset \right) \wedge \left(\forall j \in J_0 \setminus \mathrm{N}(\dot{q}),\ \left| N_j \right| = 0 \right),$$

these are:

1) $\mathrm{N}(\dot{q}) = \{2\},\ |N_1| = 0\ (\dot{\beta} = 0,\ \dot{x} \neq 0)$;
2) $\mathrm{N}(\dot{q}) = \{1\},\ |N_2| = 0\ (\dot{\beta} \neq 0,\ \dot{x} = 0)$;
3) $\mathrm{N}(\dot{q}) = \emptyset,\ |N_1| = 0,\ |N_2| = 0\ (\dot{\beta} \neq 0,\ \dot{x} \neq 0)$.

It may readily be seen that functions $|N_1|$ and $|N_2|$ do not turn zero simultaneously under any conditions, and so, case 3 is hardly ever possible.

Consider cases 1 and 2.

1) If $D^+\dot{\beta} = 0$ then from conditions $|N_1| = 0$, $\dot{\beta} = 0$ we obtain $mg\cos\alpha = 0$, what is impossible (because $0 \le \alpha < \pi/2$). Consequently, $V_2 = 0$, $D^+V_2 \neq 0$. (This is how function V_2 is determined).

2) If $D^+\dot{x} \neq 0$ then $V_1 = 0\ (\dot{x} = 0)$ и $D^+V_1 \neq 0$. (This is how function V_1 is determined).

Let $D^+\dot{x} = 0$. Hence from the condition $|N_2| = 0$ we obtain:

$$\begin{aligned} & r\ddot{\beta}\cos\beta - r\dot{\beta}^2\sin\beta + g\sin\alpha = 0 \\ & r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g\cos\alpha = 0 \end{aligned} \tag{45}$$

Having multiplied the first one of equalities (45) by $\sin\beta$, and the second one – by $\cos\beta$, and then subtracting the first one from the second one, we obtain $r\dot{\beta}^2 + g\cos(\alpha+\beta) = 0$. Consequently, $V_3 = 0$.

From the second equation of (5.17) under the condition that $D^+\dot{x} = \ddot{x} = 0$, $|N_2| = 0$ we obtain $\ddot{\beta} = -m_2 gr\sin(\alpha+\beta)/J$, whence we have $D^+V_3 = -\dot{\beta}g\sin(\alpha+\beta)(2r^2 m_2/J + 1)$.

Since in case 2 under scrutiny it true that $\dot{\beta} \neq 0$, we have $D^+V_3 \neq 0$ when

$\sin(\alpha+\beta)\neq 0$ (This is how function V_3 is determined).

If, otherwise, $\sin(\alpha+\beta)=0$ then $V_4=0$ and $D^+V_4=\cos(\alpha+\beta)\dot{\beta}\neq 0$ (This is how function V_4 is determined).

This completes investigation of conditions of Theorem 2.3.1 for the equation (5.17), what allows one to conclude on the dichotomous character of the system (this follows from Theorem 5.1.3).

The following two remarks may be expedient:

1) If $\mathrm{tg}\alpha > f_1$ then $M=\emptyset$, and there are no bounded solutions in system (5.17) (to be exact, all the solutions are infinitely large);

2) If $\mathrm{tg}\alpha = f_1$ then $M\neq\emptyset$, but equations (5.17) have a solution (an unbounded one) $x=\dot{x}_0 t + x_0$, $\dot{x}=\dot{x}_0$, $\beta=\beta_0$ $\dot{\beta}=0$, where $f_2\geq r\left|\sin(\alpha+\beta_0)\right|$, $\dot{x}_0<0$, which does not tend to M even weakly.

5.3. Stability of the set equilibrium positions for the systems with friction.

Consider the system of equations

$$A(q)\ddot{q}=g(q,\dot{q})+Q^A(q,\dot{q})+Q^T(q,\dot{q},\ddot{q}) \tag{46}$$

The set of all equilibrium positions of system (46) is defimned by the equality

$$\begin{gathered} M=\{(q,0)\in\Omega: \\ \left|g_s(q,0)+Q_s^A(q,0)\right|\leq f_s(q^s,0)\left|N_s(q,0,0)\right|,\quad s=1,\dots,k_*; \\ g_s(q,0)+Q_s^A(q,0)=0,\quad s=k_*+1,\dots,k\}. \end{gathered} \tag{47}$$

For every point $z=(q,\dot{q})\in M$ the cone $\Gamma(z)$, which defined Γ- sectors, is given in its explicit form and represents a set of vectors $(q',\dot{q}')$ such that for each $s\in(1,\dots,k_*)$ the following conditions

1) $\dot{q}'^s=0$, when $f_s(q_s,0)\left|N_s(q,0,0)\right|>\left|g(q,0)+Q^A(q,0)\right|$;

2) $\dot{q}'^s Q_s^{T0}\leq 0$, when $f_s(q^s,0)\left|N_s(q,0,0)\right|\leq\left|g_s(q,0)+Q_s^A(q,0)\right|$, $\left|N_s(q,0,0)\right|\neq 0$

hold.

Obviously, for each $z\in M$ the inclusion $M\subset\Gamma(z)$ is valid.

For those indicies $s\in(1,\dots,k_*)$, for which the condition

$\left| N_s(q, 0, 0) \right| \neq 0$ is satisfied on set M, within the Γ-sector the respective coordinates of generalized velocities $\dot{q}^s$ either turn zero or acquire a definite sign. So, defined is either some relative equilibrium or the direction of motion of system (46) with respect to the generalized coordinate $\dot{q}^s$ in the vicinity of the set of equilibrium positions (for both stable equilibrium and unstable equilibrium). The latter may relieve the difficulties bound up with investigation of signdefiniteness of the derivative $D^{*+}V$ due to system (46) because this derivative is given inexplictly. This may also turn out to be useful in investigations of not only ordinary stability, but also asymptotic stability and instability with the use the principle of invariance.

Note also that for any equilibrium position $x = (q, 0) \in M$, for which $\left| g_s + Q_s^A \right| < f_s \left| N_s \right|_{\ddot{q}=0}$ for all $s = 1, \ldots, k_*$ under the condition that $k = k_*$ within the Γ-sector, only steady motions are possible. Whence we can conclude on stability (pointwise stability) of each such equilibrium (these issues are discussed in detail below).

Theorem on stability from Section 1.3 may be reformulated to be applied to system (46) and to Γ-sectors, generated by sets $\Gamma(z)$ described above, with the nodes $z = (q, 0) \in M$. The sets of equilibrium positions of scrutinized mechanical systems are characterized by the specificity, in view of which we have to propose the following two additional theorems on asymptotic stability and on instability.

Some compact set M of equlibria for system (46) will be called isolated if there exists a number $\rho > 0$ such that its neighborhood M^ρ does not conyain any equilibrium positions, which do not belong to M.

Theorem 5.3.1. *Let M be some stable compact and isolated set of equilibrium positions related to equation (46). Suppose that there are locally Lipschitz functions $V_i(x)$, $i = 1, \ldots, m$, defined on set M^ρ, $\rho > 0$, and such that for any Γ-sector $\Omega_\delta(z)$ with the node $z \in M$ and with the radius $\delta \in (0, \rho)$ the conditions*

1) $D^{*+}V_i(z') \leq 0$ for all $z' \in \Omega_\delta(z)$, $i = 1, \ldots, m$;

2) $E \subset \{z : z = (q, \dot{q}),\ \dot{q} = 0\}$,

where $E\ @\left\{\bigcap E\left(D^{*+}V_i = 0\right) : i = 1, \ldots, m\right\}$, hold.

Hence M is asymptotically stable.

Theorem 5.3.2. *Let M be some compact and isolated set of equilibrium positions related to equation (46). Suppose that there are locally Lipschitz functions $V_i(x)$, $i = 1, \ldots, m$, defined on set M^ρ, $\rho > 0$, and such that for any Γ-sector $\Omega_\delta(z)$ with the node $z \in M$ and with the radius $\delta \in (0, \rho)$ the conditions*

1) $D^{*+}V_1(z') \geq 0$ *u* $D^{*+}V_i(z') \leq 0$ *for all* $z' \in \Omega_\delta(z)$, $i = 1, \ldots, m$ *;*

2) $E \subset \{z : z = (q, \dot{q}),\ \dot{q} = 0\}$*;*

3) $M \subset E(V_1 \leq 0)$ *and for any* $\eta > 0$ *there exists a point* $z = (q, 0) \in M^\eta \setminus M$ *such that* $V(z) > 0$ *;*

hold.

Hence set M *is unstable.*

5.4. Example. A pendulum system with friction if the hinge and in the sliding-contact bearing under the effect of elastic force.

Consider a flat mechanical system, which consists of a piston B of mass m_1 moving with friction along a horizontal straight tube Ox and is considered as a material point with the coordinate $x = q^1$, and a heavy perfectly rigid body of mass m_2 revolving with friction around a cylindric hinge, installed on the piston and placed at a distance r from the hinge to the mass center C. J_C is the inertia moment with respect to the mass center (Fig. 2). Angle β of deviation of BC from the normal to Ox, which is directed downwards, is assumed to be q^2. It is presumed that along Ox there acts an elastic force of the spring having the elasticity coefficient c, and point of the strained state is $x = 0$. The friction coefficients – f_1 in the piston and f_2 in the hinge – are assumed to be constant, $m = m_1 + m_2$, $J = J_C + m_2 r^2$.

The equations of system's motion given in Lagrange form write:

$$\begin{cases} m\ddot{x} + m_2 r \cos\beta \ddot{\beta} = m_2 r \dot{\beta}^2 \sin\beta - cx + Q_1^T \\ m_2 r \cos\beta \ddot{x} + J\ddot{\beta} = -m_2 g r \sin\beta + Q_2^T \end{cases} \tag{48}$$

The generalized friction forces are determined by formula (24) for $s = 1, 2$ $s = 1,2$, where

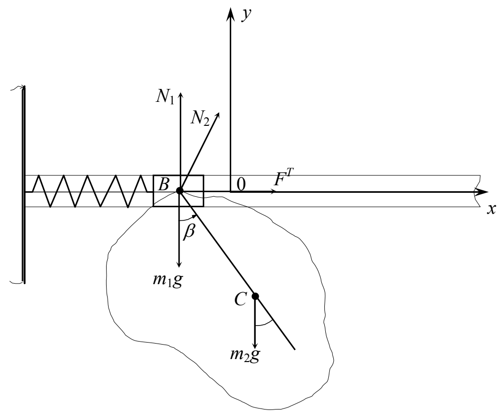

Figure 2. The pendulum system with friction in the bearing and in the hinging, which is under the effect of elastic force

$$
\begin{aligned}
&|\, N_1 \,| = \left|\, m_2 r\left(\ddot{\beta}\sin\beta + \dot{\beta}^2\cos\beta\right) + mg \,\right|, \\
&|\, N_2 \,| = m_2\left[\left(\ddot{x} + r\ddot{\beta}\cos\beta - r\dot{\beta}^2 sin\beta\right)^2 + \left(r\ddot{\beta}\sin\beta + r\dot{\beta}^2\cos\beta + g\right)^2\right]^{1/2}, \\
&Q_1^{T0} = m_2 r\left(\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta\right) + cx \ \left(\dot{x} = 0,\ \ddot{x} = 0\right), \\
&Q_2^{T0} = m_2 r\left(\ddot{x}\cos\beta + g\sin\beta\right) \ \left(\dot{\beta} = 0,\ \ddot{\beta} = 0\right).
\end{aligned}
$$

Inequalities (5.20) represent the sufficient conditions of solvability of equations (48) with respect to $\ddot{q} = \left(\ddot{x},\ \ddot{\beta}\right)$. These also are the sufficient conditions of equations (5.17).

The set of equilibrium positions for the system (48) has the form:

$$M = \left\{\left(q,\ \dot{q}\right) : \dot{x} = 0,\ \dot{\beta} = 0,\ f_1 mg \geq c\,|\,x\,|,\ f_2 \geq r\,|\sin\beta\,|\right\}.$$

Let us assume that $f_2/r < 1$. In this case, M may be considered as a set rectangles on the plane $\left(x,\ \beta\right)$. Now let us determine β_{lz} from the conditions $\sin\beta_{lz} = f_2/r$, $0 < \beta_{lz} < \pi/2$ and put $x_{lz} = f_1 mg/c$. The set $M_{lz} @ \left\{\left(q,\ \dot{q}\right) \in M : |\,x\,| \leq x_{lz},\ |\,\beta\,| \leq \beta_{lz}\right\}$ will be called the lower zone of stagnation.

Introduce the denotation:

$$
W_1 = \begin{cases} c\left(x^2 - x_{lz}^2\right)/2, & |\,x\,| > x_{lz}; \\ 0, & |\,x\,| \leq x_{lz}; \end{cases}
$$

$$W_2 = \begin{cases} m_2 gr\left(\cos\beta_{lz} - \cos\beta\right), & |\beta| > \beta_{lz}; \\ 0, & |\beta| \le \beta_{lz}; \end{cases}$$

$$T = \frac{1}{2}\left(m\dot{x}^2 + 2m_2 r\dot{x}\dot{\beta}\cos\beta + J\dot{\beta}^2\right),$$

$$V = T + W_1 + W_2,$$

$$w_1 = f_1 |N_1||\dot{x}|,\ w_2 = f_2 |N_2||\dot{\beta}|.$$

Function V is positive definite with respect to set M_{lz} in its sufficiently small neighborhood.

Let us describe sets Γ for the points $\left(x_0,\ \beta_0,\ \dot{x}_0,\ \dot{\beta}_0\right) = \left(q_0,\ \dot{q}_0\right) \in M_{lz}$ and the value of the right derivative of D^+V due to system (48) on each Γ-sector $\Omega_\delta\left(q_0,\ \dot{q}_0\right)$. First of all, note that under the condition $\left(q_0,\ \dot{q}_0\right) \in M_{lz}$ the equalities $\dot{q}_0 = 0$, $\ddot{q}_0 = 0$ и $\left|N_1\right| = mg$, $\left|N_2\right| = m_2 g$, $\left|Q_1^{T0}\right| = c\left|x\right|$, $\left|Q_2^{T0}\right| = m_2 rg\left|\sin\beta\right|$ hold.

Taking account of the fact that under the condition $\left|\beta_0\right| = \beta_{lz}$ the signs of β and $\sin\beta$ coincide within a sufficiently small neighborhood of point β_0, consider the following possible cases (the values of D^+V corresponds to points $\left(x,\ \dot{x},\ \beta,\ \dot{\beta}\right) = \left(q,\ \dot{q}\right) \in \Omega_\delta\left(q_0,\ \dot{q}_0\right)$):

1) $\left|x_0\right| < x_{lz}$, $\left|\beta_0\right| < \beta_{lz}$ ($\left(q_0,\ \dot{q}_0\right)$ is the internal point of the rectangle M_{lz}). Hence

$$\Gamma\left(q_0,\ \dot{q}_0\right) = \left\{\left(q,\ \dot{q}\right) : \dot{x} = 0,\ \dot{\beta} = 0\right\},\ D^+V = 0;$$

2) $\left|x_0\right| < x_{lz}$, $\left|\beta_0\right| = \beta_{lz}$ or $\left|\beta_0\right| < \beta_{lz}$, $\left|x_0\right| = x_{lz}$ (sides of the rectangle M_{lz} without nodes). Hence

$$\Gamma\left(q_0,\ \dot{q}_0\right) = \left\{\left(q,\ \dot{q}\right) : \dot{x} = 0,\ \dot{\beta}\beta_0 \le 0\right\},$$

$$D^+V = \begin{cases} -w_2, & |\beta| > \beta_{lz}; \\ -w_2 + m_2 gr\left|\sin\beta\right|\left|\dot\beta\right|, & |\beta| \le \beta_{lz}; \end{cases}$$

or, respectively, $\Gamma\left(q_0,\, \dot q_0\right) = \left\{\left(q,\, \dot q\right): \dot\beta = 0,\ \dot x x_0 \le 0\right\}$,

$$D^+V = \begin{cases} -w_1, & |x| > x_{lz}; \\ -w_1 + c|x||\dot x|, & |x| \le x_{lz}; \end{cases}$$

3) $|x_0| = x_{lz}$, $|\beta_0| = \beta_{lz}$ (nodes of the rectangle M_{lz}). Hence

$$\Gamma\left(q_0,\, \dot q_0\right) = \left\{\left(q,\, \dot q\right): \dot\beta\beta_0 \le 0,\ \dot x x_0 \le 0\right\},$$

$$D^+V = \begin{cases} -w_1 - w_2, & |\beta| > \beta_{lz},\ |x| > x_{lz}; \\ -w_1 - w_2 + c|x||\dot x|, & |x| \le x_{lz},\ |\beta| > \beta_{lz}; \\ -w_1 - w_2 + m_2 gr|\sin\beta||\dot\beta|, & |\beta| \le \beta_{lz},\ |x| > x_{lz}; \\ -w_1 - w_2 + c|x||\dot x| + m_2 gr\left|\sin\beta\right|\left|\dot\beta\right|, & |\beta| \le \beta_{lz},\ |x| \le x_{lz}; \end{cases}$$

Hence we have 9 possible forms (kinds) of sets Γ, and within each Γ-sector the generalized velocities $\dot x$, $\dot\beta$ either turn zero or retain their signs, which are opposite to signs of x_0 and β_0, respectively.

In case 1, sign definiteness of $D^+V\left(q,\, \dot q\right)$ does not need any further analysis. In cases 2 and 3, the sign of D^+V is determined by relations between the values of

functions $f_1|N_1|$ and $c|x|$, $f_2|N_2|$ and $m_2 gr|\sin\beta|$ on set $\Gamma\left(q,\, \dot q\right)$. It can readily be noticed that the condition $D^+V \le 0$ holds (within the Γ-sector) when for any point $\left(q_0,\, \dot q_0\right) \in M_{lz}$ along each solution of equations (48) with the values lying in the Γ-sector $\Omega_\delta\left(q_0,\, \dot q_0\right)$ the inequality

$$D^+\dot\beta\sin\beta + \dot\beta^2\cos\beta \ge 0 \qquad (49)$$

holds.

Indeed, in this case, from the inequality $|x| \le x_{lz}$ we have

$f_1 | N_1 | \geq f_1 mg \geq c | x |$ and from the inequality $| \beta | \leq \beta_{lz}$ we have $f_2 | N_2 | \geq f_2 m_2 g \geq m_2 gr | \sin \beta |$, whence, on account of the form of $D^+ V$, we obtain that $D^+ V \leq 0$.

In order to prove (49), *suppose the opposite*. Since function $D^+ \dot{\beta}(t)$ is right continuous (i.e. the solution if R- right), the following inequality is satisfied

$$D^+ \dot{\beta} \sin \beta + \dot{\beta}^2 \cos \beta < 0 \tag{50}$$

on some small segment $[0, \alpha)$. By integrating (50), we obtain that $\dot{\beta}(t) \sin \beta(t) - \dot{\beta}(0) \sin \beta(0) < 0$ for all $t \in (0, \alpha)$. If $| \beta_0 | < \beta_{lz}$ then $\dot{\beta}(t) = 0$ for sufficiently small $t > 0$ and, consequently, (49) holds. Therefore, (50) will be satisfied only under the condition that $| \beta_0 | = \beta_{lz}$. Hence it is possible to assume that $\sin \beta(t) \neq 0$. Let for the purpose of definiteness $\sin \beta(t) > 0$. Hence $\dot{\beta}(t) \leq 0$ and, consequently, $\sin \beta(t)$ is a nonincreasing function. Therefore, the inequality $\sin \beta(t) \leq \sin \beta(0)$ holds, and so, on account of the condition $\dot{\beta}(t) \leq 0$, we have $\dot{\beta}(0) \sin \beta(t) \geq \dot{\beta}(0) \sin \beta(0) \geq \dot{\beta}(t) \sin \beta(t)$.

Consequently, $\dot{\beta}(0) \geq \dot{\beta}(t)$, whence we obtain $D^+ \dot{\beta}(0) \geq 0$. But then inequality (50) is not satisfied for $t = 0$, what contradicts to the above assumption.

The case when $\sin \beta(t) < 0$ is considered similarly and also leads to the contradiction with (50).

Therefore, all the conditions of the theorem on stability, according to which the lower zone of stagnation is stable, are satisfied for equations (50), for the set M_{lz} and for the function V.

In order to investigate asymptotic stability of M_{lz} with the aid of Theorem 5.3.1, consider the functions $V_1 = x^2/2$, $V_2 = \beta^2/2$.

Hence for any $(q_0, \dot{q}_0) \in M_{lz}$ and for Γ- sectors $\Omega_\delta(q_0, \dot{q}_0)$ the conditions

$$D^+ V_1 = x\dot{x} \leq 0, \quad D^+ V_2 = \beta\dot{\beta} \leq 0.$$

are satisfied. If either $x_0 = 0$ or $\beta_0 = 0$ then for a sufficiently small $\delta > 0$ for all

$(q, \dot{q}) \in \Omega_\delta (q_0, \dot{q}_0)$ it is true that either $\dot{x} = 0$ or, respectively, $\dot{\beta} = 0$. If $x_0 \neq 0$ and $\beta_0 \neq 0$ then

$$E\left(D^+ V_1 = 0\right) \cap E\left(D^+ V_2 = 0\right) = \left\{(q, \dot{q}) : \dot{x} = 0,\ \dot{\beta} = 0\right\}.$$

Therefore, the conditionвсегда выполняется

$$\Omega_\delta (q_0, \dot{q}_0) \cap E\left(D^+ V_1 = 0\right) \cap E\left(D^+ V_2 = 0\right) \subset \left\{(q, \dot{q}) : \dot{x} = 0,\ \dot{\beta} = 0\right\}$$

does not hold always, and, according to Theorem 3.1, M_{lz} is asymptotically stable.

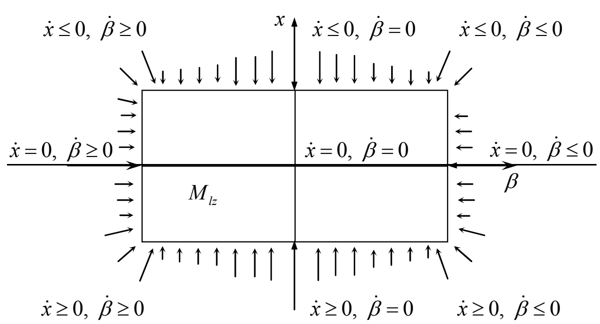

Figure 3. The lower zone of stagnation

Note in conclusion that usage of Γ- sectors allows one to give explicit geometric interpretations of behavior of the motions in the vicinity of set M of equilibria (both stable and unstable ones). This is because the behavior of trajectories within the Γ- sector may be substantially simplified. As far as system (48) is concerned, the phase space is represented by the 4-dimensional space of variable $(x, \beta, \dot{x}, \dot{\beta})$. Nevertheless, Fig.3 gives a sufficiently complete idea of behavior of the trajectories near the lower zone of stagnation. As regards Γ- sectors with the nodes inside the rectangle M_{lz}, only steady motions are possible.

5.5. Pointwise stability of equilibrium positions.

Pointwise stability of internal equilibriua for dynamics equations is discussed herein. Such stability may be ensured by the structure of equations themselves, without any additional assumptions.

Consider equations of dynamics (24) in the autonomous case. Introduce the denotation $Q_s(q, \dot{q}) = g_s(q, \dot{q}) + Q_s^A(q, \dot{q}),\ s = 1, \ldots, k$.

D e f i n i t i o n . *The equilibrium position* $(q, 0)$, *для которого*

$f_s \left| N_s \right| > \left| Q_s \right|$ *for all* $s = 1, \ldots, k_*$*, называется внутренним. Множество всех внутренних положений равновесия обозначается* M^0.

Introduce the following denotations:

$$\dot{q}^{0*} = (\underbrace{0, \ldots, 0}_{k_*}, \dot{q}^{k_*+1}, \ldots, \dot{q}^k), \ \dot{q}^* = (\dot{q}^{k_*+1}, \ldots, \dot{q}^k).$$

Having put $\ddot{q}^s = 0$, $s = 1, \ldots, k_*$, in the equations of dynamics, having rejected the first three groups of equations and added the conditions $\dot{q}^s = 0$, $s = 1, \ldots, k_*$, to the 4th group, we obtain the following relations

$$\begin{cases} \dot{q}^s = 0 \left(s = 1, \ldots, k_* \right), \\ \sum_{i=k_*+1}^{k} a_{si}\left(q\right)\ddot{q}^i = Q_s\left(q, \dot{q}^{0*}\right) \left(s = k_* + 1, \ldots, k\right). \end{cases} \tag{51}$$

Let us consider (51) as a system of differential equations with phase variables in space R^{2k-k_*} /In order to avoid any changes in denotations and not to make emphasis on the relationship with equations of dynamics, let us take $\left(q, \dot{q}^*\right)$ in the capacity of such equations. The points $\left(q, 0^*\right) @ (q^1, \ldots, q^k, \underbrace{0, \ldots, 0}_{k-k_*})$, where q satisfies equalities $Q_s\left(q, 0\right) = 0, \left(\left(s = k_* + 1, \ldots, k\right)\right.$, will be considered as equilibrium positions for the system (53).

The set of all equilibrium positions corresponding to equations (51) is denoted by M^*. If $\left(q_0, 0\right)$ is an equilibrium position corresponding to the equations of dynamics, then $\left(q_0, 0^*\right)$ is *the corresponding equilibrium position for equations (*51*).* Pointwise stability of equilibrium positions $\left(q_0, 0\right) \in M$ is understood in the general sense.

D e f i n i t i o n . *Let us speak that the equilibrium position* $\left(q_0, 0\right) \in M$ *is strongly asymptotically stable with respect to variables* $\dot{q}^i$, $i = 1, \ldots, k_*$*, if for any* $\varepsilon > 0$ *and* $\tau > 0$ *there exists* $\delta > 0$ *such that any solution* $\left(q\left(t\right), \dot{q}\left(t\right)\right)$ *with the initial condition* $\left(q\left(0\right), \dot{q}\left(0\right)\right) \in S_\delta\left(q_0, 0\right)$ *exists and satisfies the condition* $\left(q\left(t\right), \dot{q}\left(t\right)\right) \in S_\varepsilon\left(q_0, 0\right)$ *for all* $t \geq 0$ *and* $\dot{q}^i\left(t\right) = 0$ *for all* $t \geq \tau$, $i = 1, \ldots, k_*$.

T h e o r e m 5 . 5 . 1 . *The internal equilibrium position* $(q_0, 0) \in M^0$ *is stable if and only if the respective equilibrium position* $(q_0, 0^*) \in M^*$ *is stable.*

C o r o l l a r y 5 . 5 . 1. *If* $k = k_*$ *in the dynamics equations then any internal equilibrium position* $(q_0, 0)$ *is strongly asymptotically stable with respect to variables* $\dot{q}^1, \ldots, \dot{q}^{k_*}$.

Below we assme that $k = k_*$. By M^0_{top} and ∂M we denote, respectively, the topological interior and the boundary of set M with respect to the subspace $L @ \{(q, \dot{q}) : \dot{q}^s = 0,\ s = 1, \ldots, k\}$.

T h e o r e m 5 .5.2. *If* $M^0_{top} \neq \emptyset$ *and* $M^0_{top} \subset M^0$ *then* $M^0_{top} = M^0$*, and any compact set* $K \subset M^0_{top}$ *is stable. Furthermore, for each* $\tau > 0$ *there exists* $\delta > 0$ *such that for any solution* $z(t) = (q(t), \dot{q}(t))$ *of the dynamics equations with the initial condition* $z(0) \in K^{\delta_0}$ *the inclusion* $z(t) \in M^0_{top}$ *hold for all* $t \geq \tau$.

T h e o r e m 5 .5.3. *Let set* $M \subset \Omega$ *be compact,* $M^0_{top} \neq \emptyset$ *u* $M^0_{top} \subset M^0$. *Hence* $M^0_{top} = M^0$*, and set* M *is stable if and only if* ∂M *is stable.*

In case of P. Painlevé's example described above, the set equilibrium positions writes^ $M = \{(x, \theta, 0, 0) : \cos\theta = 0\}$. Since for each point $(x, \theta, 0, 0) \in M$ the conditions $f_1 | N_1 | = 2 f_1 g > 0$ and $Q_1^{T0} = 0$ hold, in accordance with the accepted definition, all the equilibrium positions from M are internal. System (51) assumes the form:

$$\begin{cases} \dot{x} = 0, \\ r\ddot{\theta} = g\cos\theta \end{cases} \tag{52}$$

As far as system (52) is concerned, the set equilibria M^* consists of the points $(x, \theta, 0)$, for which $\cos\theta = 0$.

Note in conclusion, the basis of the present paper has been formed by publications of the authors [34]-[44]. Note also that these investigations were later presented in papers [45] – [46], in which amore general class of equations of motion for mechanical systems with Coulomb's friction was considered. Nevertheless, existence of right solutions was not proved in above papers, and the solution itself is defined with the use of some modification in the obtaining solutions of discontinuous systems in the sense of A.F. Filippov.

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